The fundamental unit of some quadratic, cubic or quartic orders

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Abstract. We explain why it is reasonable to conjecture that if \( \epsilon \) is a totally imaginary quartic unit, then \( \epsilon \) is in general a fundamental unit of the quartic order \( \mathbb{Z}[\epsilon] \), order whose group of units is of rank equal to one. We partially prove this conjecture. This generalizes a result of T. Nagell, who proved in 1930 a similar result for real cubic units with two non real conjugates.


1. Introduction

Let \( \epsilon \) be an algebraic unit. Assume that the group of units of the order \( \mathbb{Z}[\epsilon] \) is of rank equal to 1. It is a natural question to ask whether \( \epsilon \) is a generator of this unit group. We must assume that \( \epsilon \) is not a complex root of unity, otherwise, being of finite order, it cannot be a generator of the infinite group of units of the order \( \mathbb{Z}[\epsilon] \).

Clearly, \( K_\epsilon = \mathbb{Q}(\epsilon) \) must be a real quadratic number field, in which case we may assume that \( \epsilon > 1 \), or must be a cubic number field with one real embedding and two non-real complex embeddings, in which case we may assume that \( \epsilon \) is real and that \( \epsilon > 1 \), or must be a totally imaginary quartic number field, in which case we may assume that \( |\epsilon| > 1 \) ([8, Lemma 1.6]).

In these quadratic and cubic situations, the only complex roots of unity contained in \( \mathbb{Z}[\epsilon] \) are \(+1\) and \(-1\), and there exists only one unit \( \eta > 1 \) of \( \mathbb{Z}[\epsilon] \) such that the unit group of the order \( \mathbb{Z}[\epsilon] \) is \( \{\pm \eta^n; \ n \in \mathbb{Z}\} \), this unit
being called the fundamental unit of this order. In both these situations, there exists some \( n \geq 1 \) such that \( \epsilon = \eta^n \), which implies \( Z[\epsilon] = Z[\eta^n] = Z[\eta] \) and \( d_\epsilon = d_\eta \) (absolute values of discriminants of minimal monic polynomials).

In Theorem 1, we will prove that, in the quadratic case, \( \epsilon \) is always a fundamental unit of the quadratic order \( Z[\epsilon] \), with one exception. In Theorem 3, we will prove that, in the cubic case, \( \epsilon \) is always a fundamental unit of the cubic order \( Z[\epsilon] \), with a one parameter infinite family of exceptions where \( \epsilon \) is a square of a fundamental unit, together with eight sporadic exceptions. In Conjecture 8, we will conjecture that, in the quartic case, \( \epsilon \) is always a fundamental unit of the quartic order \( Z[\epsilon] \), with a one parameter infinite family of exceptions where \( \epsilon \) is a square of a fundamental unit, together with fourteen sporadic exceptions. We will not be able to fully prove this conjecture, but at least we will prove in Theorem 10 that it holds true if either (i) the quartic field \( Q(\epsilon) \) contains an imaginary quadratic subfield, or (ii) if the quartic order \( Z[\epsilon] \) contains non trivial (i.e. \( \neq \pm 1 \)) complex roots of unity. We will also add some compelling arguments making this conjecture very reasonable. Finally, we will explain what we lack to prove this conjecture: a result on quartic polynomials (see Conjecture 22) which we can prove for quadratic polynomials (see (1) in Theorem 1) and for cubic polynomials (see (3) in Theorem 4) and that we used in Theorems 1 and 3 to settle our problem in the quadratic and cubic cases. In the quartic case, we can only prove the easy part of this Conjecture 22 (see (8)).

2. The quadratic case

The quadratic case is easy:

**Theorem 1.** Let \( \epsilon > 1 \) be a real quadratic unit of discriminant \( d_\epsilon > 0 \). Then, \( \epsilon \) is the fundamental unit of the quadratic order \( Z[\epsilon] \), except if \( \epsilon = (3+\sqrt{5})/2 \), in which case \( \epsilon = \eta^2 \), where \( 1 < \eta = (1+\sqrt{5})/2 = \epsilon - 1 \in Z[\epsilon] \) is the fundamental unit of \( Z[\epsilon] = Z[\eta] \), and \( d_\epsilon = d_\eta = 5 \). Moreover, we have

\[
\epsilon^2/3 \leq d_\epsilon \leq 2\epsilon^2.
\]

**Proof.** To begin with, let \( \alpha > 1 \) be a quadratic algebraic unit, let \( \alpha' = \pm 1/\alpha \), \( \Pi_\alpha(X) = X^2 - aX \pm 1 \in Z[X] \) and \( d_\alpha = a^2 + 4 > 0 \) be its conjugate, its minimal monic quadratic polynomial and its discriminant. Then, \( \alpha = (a+\sqrt{\alpha^2})/2 \geq (1+\sqrt{5})/2 \) and \( (\alpha-1/\alpha)^2 \leq d_\alpha = (\alpha-\alpha')^2 \leq (\alpha+1/\alpha)^2 \).

Now, write \( \epsilon = \eta^n \) and assume that \( n \geq 2 \). Then,

\[
(\eta^2 - \eta^{-2})^2 \leq (\eta^n - \eta^{-n})^2 = (\epsilon - \epsilon^{-1})^2 \leq d_\epsilon = d_\eta \leq (\eta + \eta^{-1})^2.
\]

Hence, \( 0 < \eta - \eta^{-1} \leq 1 \), \( 1 < \eta \leq (1 + \sqrt{5})/2 \) and \( \eta = \eta_0 = (1 + \sqrt{5})/2 \). Then, \( (\eta_0^2 - \eta_0^{-2})^2 \leq (\eta_0 + \eta_0^{-1})^2 \) implies \( n = 2 \) and \( \epsilon = \eta_0^2 = (3 + \sqrt{5})/2 \).
For (1), we use \( \frac{1 - \sqrt{\frac{3}{2}e^2}}{2} = (1 - 1/\eta_0^2)^2e^2 \leq (1 - 1/e^2)^2e^2 \leq d_e \leq (1 + 1/e^2)^2e^2 \leq (1 + 1/\eta_0^2)^2e^2 = 5\frac{3 - \sqrt{3}}{2}e^2. \)

The idea used in this proof will be used to prove Proposition 12.

3. The cubic case

Definition 2. A cubic polynomial of type (T) is a monic cubic polynomial \( P(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X] \) which is \( \mathbb{Q} \)-irreducible (\( \leftrightarrow b \neq a \) and \( b \neq -a - 2 \)), of negative discriminant \( -d_{P(X)} < 0 \), with \( d_{P(X)} = 4(a^3 + b^3) - a^2b^2 - 18ab + 27 > 0 \), and whose only real root \( \epsilon_P \) satisfies \( \epsilon_P > 1 \) (\( \Leftrightarrow P(1) < 0 \Leftrightarrow b \leq a - 1 \)). In that case, we have (see [1, Lemma 1]):

\[ |b| < 1 + 2\sqrt{a+2} \quad \text{and} \quad 0 \leq a \leq \epsilon_P + 2. \]  

Hence, we can sort the cubic polynomials of type (T) by increasing values of \( a \geq 0 \).

A non totally real algebraic cubic unit \( \epsilon > 1 \) is the real root of a monic cubic polynomial \( \Pi_3(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X] \) of type (T), its minimal monic polynomial. Set \( d_e = |d_{\Pi_3(X)}| > 0 \). If \( \epsilon = \eta^n \) for some \( n \geq 2 \) and some algebraic unit \( \eta \in \mathbb{Z}[\epsilon] \) satisfying \( \eta > 1 \), then \( \mathbb{Z}[\epsilon] = \mathbb{Z}[\eta] \), hence \( d_e = d_\eta \), and \( \Pi_3(X) \) and \( \Pi_\eta(X) \) are two cubic polynomials of type (T) of the same discriminant. In 1930, T. Nagell proved the following result:

Theorem 3 (See [2]). Let \( \epsilon > 1 \) be a real algebraic cubic unit of negative discriminant \( -d_e < 0 \). Let \( \eta > 1 \) be the fundamental unit of the cubic order \( \mathbb{Z}[\epsilon] \). Then, \( \epsilon = \eta \), except in the following cases:

1. The infinite family of exceptions for which \( \epsilon \) is the only real root of \( X^3 - M^2X^2 - 2MX - 1, M \geq 1 \), in which case \( \epsilon = \eta^2 \) where \( \eta = \epsilon^2 - M^2 \epsilon - M \in \mathbb{Z}[\epsilon] \) is the real root of \( X^3 - MX^2 - 1, \mathbb{Z}[\eta] = \mathbb{Z}[\epsilon] \) and \( d_e = d_\eta = 4M^3 + 27 \).

2. The 8 following sporadic exceptions:

   (a) i. \( \Pi_3(X) = X^3 - 2X^2 + X - 1 \), in which case \( \epsilon = \eta^2 \) where \( \eta = \epsilon^2 - \epsilon \in \mathbb{Z}[\epsilon] \).
   (b) i. \( \Pi_3(X) = X^3 - 3X^2 + 2X - 1 \), in which case \( \epsilon = \eta^3 \) where \( \eta = \epsilon - 1 \in \mathbb{Z}[\epsilon] \).
   (c) i. \( \Pi_3(X) = X^3 - 2X^2 - 3X - 1 \), in which case \( \epsilon = \eta^4 \) where \( \eta = \epsilon^2 - 2\epsilon - 2 \in \mathbb{Z}[\epsilon] \).
   (d) i. \( \Pi_3(X) = X^3 - 5X^2 + 4X - 1 \), in which case \( \epsilon = \eta^5 \) where \( \eta = \epsilon^2 - 4\epsilon + 1 \in \mathbb{Z}[\epsilon] \).
v. $\Pi_1(X) = X^3 - 12X^2 + 7X - 1$, in which case $\epsilon = \eta^9$ where $\eta = -3\epsilon^2 + 37\epsilon + 10 \in \mathbb{Z}[\epsilon].$

In these five cases, $\eta > 1$ is the real root of $\Pi_\eta(X) = X^3 - X - 1$, $\mathbb{Z}[\eta] = \mathbb{Z}[\epsilon]$ and $d_\eta = d_\epsilon = 23.$

(b) i. $\Pi_\epsilon(X) = X^3 - 4X^2 + 3X - 1$, in which case $\epsilon = \eta^3$ where $\eta = \epsilon^2 - 3\epsilon + 1 \in \mathbb{Z}[\epsilon].$

ii. $\Pi_\epsilon(X) = X^3 - 6X^2 - 5X - 1$, in which case $\epsilon = \eta^5$ where $\eta = -2\epsilon^2 + 13\epsilon + 5 \in \mathbb{Z}[\epsilon].$

In these two cases, $\eta > 1$ is the real root of $\Pi_\eta(X) = X^3 - X^2 - 1$, $\mathbb{Z}[\eta] = \mathbb{Z}[\epsilon]$ and $d_\eta = d_\epsilon = 31.$

(c) $\Pi_\eta(X) = X^3 - 7X^2 + 5X - 1$, in which case $\epsilon = \eta^3$, where $1 < \eta = -\epsilon^2 + 7\epsilon - 3 \in \mathbb{Z}[\epsilon]$ is the real root of $\Pi_\eta(X) = X^3 - X^2 - X - 1$, $\mathbb{Z}[\eta] = \mathbb{Z}[\epsilon]$ and $d_\eta = d_\epsilon = 44.$

In 2006 we gave a new proof of Theorem 3 by using the following Theorem 4, the following Lemma 5, and numerical computations and the following Lemma 6 to deal with the finitely many cases where $d_\epsilon < 712.$

**Theorem 4 (See [1, Lemma 2 and Theorem 2]).** Let $\epsilon > 1$ be a real cubic algebraic unit of negative discriminant $-d_\epsilon < 0.$ Then (compare with (1)),

$$e^{3/2}/2 \leq d_\epsilon \leq 64e^3. \quad (3)$$

Hence, by (2), there are only finitely many such cubic units of bounded discriminant. Finally, if $d_\epsilon \geq 712,$ then $\epsilon$ is either the fundamental unit of the cubic order $\mathbb{Z}[\epsilon]$ or its square. In any case, $\epsilon$ is at most an eleventh power of the fundamental unit of this order.

**Proof.** In fact, we have the better upper bound $d_\epsilon \leq 4(e^{3/4} + e^{-3/4})^4$ (see [1, (2)]). Assume that $\epsilon = \eta^n$ with $1 < \eta \in \mathbb{Z}[\epsilon]$ and $n \geq 3.$ Then,

$$e^{3/2}/2 \leq d_\epsilon = d_\eta \leq 4(\eta^{3/4} + \eta^{-3/4})^4 \leq 4(e^{1/4} + e^{-1/4})^4.$$  

Hence, $\epsilon < 126.51$ and $d_\epsilon \leq 4(e^{1/4} + e^{-1/4})^4 < 712.$

For the last assertion, we first check using (2) that we always have $\epsilon \geq \epsilon_0 = 1.32471 \ldots,$ where $\epsilon_0 > 1$ is the only real root of $X^3 - X - 1$ of type (T). Hence, if $\epsilon = \eta^n$ with $n \geq 2$ and $\eta \in \mathbb{Z}[\epsilon],$ we have $\eta^{3n/2} = e^{3/2} \leq 2d_\epsilon = 2d_\eta \leq 8(\eta^{3/4} + \eta^{-3/4})^4,$

which in using $\eta \geq \epsilon_0$ yields $n \leq 11.$ \hfill \square

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1Note the misprint in [1, Theorem 4(2)] where we wrote $\epsilon = \eta^9$ instead of $\epsilon = \eta^3.$ Note also the misprint in the displayed formula in [1, Proof of Theorem 2, case (i)] which should be $d_P \geq (\epsilon_P^2 - 4\epsilon_P^{3/2} + 1)(1 - \epsilon_P^{-3/2})^4 \geq \epsilon_P^2/2$ for $\epsilon_P \geq 65.$
Lemma 5 (Compare with Lemma 16). Let \( \epsilon > 1 \) be a real cubic algebraic unit of negative discriminant \( -d_\epsilon < 0 \). Then, \( \epsilon \) is a square in \( \mathbb{Z}[\epsilon] \) if and only if we are in one of the following three cases:

1. \( \Pi_0(X) = X^3 - M^2X^2 - 2MX - 1 \) with \( M \geq 1 \), in which case \( \epsilon = \eta^2 \) where \( \eta = \epsilon^2 - M^2\epsilon - M \in \mathbb{Z}[\epsilon] \), \( \Pi_0(X) = X^3 - M^2X^2 - 1 \) and \( d_\eta = d_\epsilon = 4M^3 + 27 \).
2. \( \Pi_0(X) = X^3 - 2X^2 - 3X - 1 \), in which case \( \epsilon = \eta^2 \) where \( \eta = -\epsilon^2 + 3\epsilon + 2 \in \mathbb{Z}[\epsilon] \), \( \Pi_0(X) = X^3 - 2X^2 + X - 1 \) and \( d_\eta = d_\epsilon = 23 \).
3. \( \Pi_0(X) = X^3 - 2X^2 + X - 1 \), in which case \( \epsilon = \eta^2 \) where \( \eta = \epsilon^2 - \epsilon \in \mathbb{Z}[\epsilon] \), \( \Pi_0(X) = X^3 - X - 1 \) and \( d_\eta = d_\epsilon = 23 \).

Proof. Assume that \( \epsilon = \eta^2 \), with \( 1 < \eta \in \mathbb{Z}[\epsilon] \). Then, \( \mathbb{Z}[\epsilon] = \mathbb{Z}[\eta] \) and \( d_\epsilon = d_\eta \). The index \( (\mathbb{Z}[\eta] : \mathbb{Z}[\eta^2]) \) is equal to \( |ab - 1| \), where \( \Pi_0(X) = X^3 - aX^2 + bX - 1 \). Hence, we must have \( |ab - 1| = 1 \), and we will have \( \eta = (\epsilon^2 - (a^2 - b)e - a)/(1 - ab) \) and \( \Pi_0(X) = X^3 - (a^2 - 2b)X^2 + (b^2 - 2a)X - 1 \).

First, assume that \( ab = 2 \). Then, \( a = 2 \) and \( b = 1 \) (for \( a \geq 0 \) and \( b \leq a - 1 \)), \( \Pi_0(X) = X^3 - 2X^2 + X + 1 \) and \( \Pi_0(X) = X^3 - 2X^2 - 3X - 1 \).

Second, assume that \( ab = 0 \). If \( a = 0 \), then \( b \leq a - 1 = -1 \) and \( d_\eta = 4b^3 + 27 > 0 \) yields \( b = -1 \), \( \Pi_0(X) = X^3 - X - 1 \), and \( \Pi_0(X) = X^3 - 2X^2 + X - 1 \). If \( a \neq 0 \), then \( b = 0 \), \( \Pi_0(X) = X^3 - aX^2 - 1 \), \( d_\eta = 4a^3 + 27 \) and \( \Pi_0(X) = X^3 - a^2X^2 - 2aX - 1 \).

Lemma 6 (Compare with Lemma 18). Let \( \epsilon > 1 \) and \( 1 < \eta \in \mathbb{Z}[\epsilon] \) be two real algebraic cubic units of negative discriminant. Write \( \Pi_0(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X] \) and \( \Pi_0(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X] \). If \( \epsilon = \eta^k \) with \( n \geq 3 \), then \( a_\epsilon > a_\eta \) and \( d_\epsilon = d_\eta \). Hence, if \( \epsilon \) is neither the fundamental unit of the order \( \mathbb{Z}[\epsilon] \) nor its square, then there exists \( P(X) = X^3 - aX^2 + bX + 1 \in \mathbb{Z}[X] \) of type (T) such that \( 0 \leq a < a_\epsilon \) and \( d_P = d_\epsilon \).

Proof. If \( \epsilon_P > 1 \) is the real root of \( P(X) = X^3 - aX^2 + bX - 1 \) of type (T) and if \( \epsilon_P \) and \( \epsilon_P^* \) are its two complex conjugate roots, then \( \epsilon_P | \epsilon_P^* |^2 = 1 \). Hence,

\[
\epsilon_P - 2/\sqrt{\epsilon_P} \leq a = \epsilon_P + 2\Re(\epsilon_P) \leq \epsilon_P + 2/\sqrt{\epsilon_P}.
\]

Suppose that \( 0 \leq a_\epsilon \leq a_\eta \). We have

\[
\eta^3 - 2/\eta^3/2 \leq \epsilon - 2/\sqrt{\epsilon} \leq a_\epsilon \leq a_\eta \leq \eta + 2/\sqrt{\eta},
\]

which implies \( \eta < 1.61 \).

Notice that [1, Lemma 3] should have been stated as follows: Let \( P(X) = X^3 - a_PX^2 + b_PX - 1 \) and \( Q(X) = X^3 - a_QX^2 + b_QX - 1 \) be two cubic polynomials of type (T). Then, \( \epsilon_P \geq \epsilon_Q^2 \) implies \( a_P > a_Q \) except in the following two cases: (i) \( Q(X) = X^3 - 2X^2 + X - 1 \) and \( P(X) = X^3 - 2X^2 - 3X - 1 \) where \( \epsilon_P = \epsilon_Q^2, a_P = a_Q = 2 \) and \( d_P = d_Q = 23 \), and (ii) \( Q(X) = X^3 - X^2 - 1 \) and \( P(X) = X^3 - X^2 - 2X - 1 \) where \( \epsilon_P = \epsilon_Q^2, a_P = a_Q = 1 \) and \( d_P = d_Q = 31 \).
Now, by (2), if \( \eta < 1.61 \), then either (i) \( \Pi_\eta(X) = X^3 - X - 1 \) with \( d_\eta = 23 \) and \( a_\eta = 0 \), or by (2) there is no cubic polynomial \( \Pi_\eta(X) \neq \Pi_\eta(X) \) of type (T) with \( a_\sigma = 0 \) and \( d_\sigma = 23 \), or (ii) \( \Pi_\eta(X) = X^3 - X^2 - 1 \) with \( d_\eta = 31 \) and \( a_\eta = 31 \), but by (2) \( \Pi_\eta(X) = X^3 - X^2 - 2X - 1 \) is the only cubic polynomial \( \Pi_\eta(X) \neq \Pi_\eta(X) \) of type (T) with \( 0 \leq a_\sigma \leq 1 \) and \( d_\sigma = 31 \) and this yields \( \epsilon = \eta^n \) with \( n = 2 \).

\[ \square \]

4. The quartic case

We would like to prove similar results for quartic orders of unit groups of rank 1. So, let \( \epsilon \) be a totally imaginary quartic algebraic unit. Let \( \Pi_\epsilon(X) = X^4 - aX^3 + bX^2 - cX + 1 \in Z[X] \) be its minimal monic polynomial. It is \( Q \)-irreducible and of positive discriminant \( d_\epsilon \). Since \( Z[\epsilon] = Z[-\epsilon] = Z[1/\epsilon] = Z[-1/\epsilon] \), we may assume that \( 0 \leq |\epsilon| \leq a \). Let \( \epsilon, \epsilon', \epsilon'' \) and \( \epsilon''' \) denote the four complex roots of \( \epsilon \). Since, \( |\epsilon'| = 1/|\epsilon| \), we may also assume that \( |\epsilon| \geq 1 \). If \( |\epsilon| = 1 \), then \( |\epsilon| = |\epsilon'| = 1, \epsilon \) is a complex root of unity (see [8, Lemma 1.6]), and being of finite order, it is not a fundamental unit of the totally imaginary quartic order \( Z[\epsilon] \). We exclude this case by defining:

**Definition 7 (Compare with Definition 2).** A quartic polynomial of type (T) is a monic quartic polynomial \( P(X) = X^4 - aX^3 + bX^2 - cX + 1 \in Z[X] \) which is \( Q \)-irreducible, satisfying \( 0 \leq |\epsilon| \leq a \), of positive discriminant \( d_\epsilon = -4a^3c^3 + a^2b^2c^2 + 18a^3bc + 18abc^3 - 4a^2b^3 + 4b^3c^2 - 27a^4 - 27c^4 - 6a^2c^2 - 80abc^2 + 16b^4 + 144a^2b - 144bc^2 - 192ac - 128b^2 + 256 > 0 \), with no real root and with at least one complex root of absolute value greater than one. In that case, for any complex root \( \epsilon_\rho \) of \( P(X) \), we have

\[ |\epsilon| \leq a \leq \sqrt{4b + 2} \quad \text{and} \quad -1 \leq b \leq |\epsilon_\rho|^2 + 1/|\epsilon_\rho|^2 + 4. \quad (4) \]

Hence, we can sort the quartic polynomials of type (T) by increasing values of \( b \geq -1 \).

**Proof.** Let \( \epsilon, \epsilon', \epsilon'' \) and \( \epsilon''' \) be the four complex roots of \( P(X) \). Then, \( a = 2R(\epsilon) + 2R(\epsilon') \) and \( b = |\epsilon|^2 + |\epsilon'|^2 + 4R(\epsilon)R(\epsilon') \). Using \( |\epsilon'| = 1/|\epsilon| \) and \( |\epsilon| > 1 \), we obtain \( a \leq 2(|\epsilon| + 1/|\epsilon|), b > -2 \) and \( b \geq |\epsilon|^2 + 1/|\epsilon|^2 - 4 \geq (a/2)^2 - 6 \). \[ \square \]

From now on, \( \epsilon \) is a totally imaginary quartic algebraic unit satisfying \( |\epsilon| > 1 \) and whose minimal monic quartic polynomial \( \Pi_\epsilon(X) \) is of type (T). We want to prove that, in general, \( \epsilon \) is a fundamental unit of the totally imaginary quartic order \( Z[\epsilon] \) (whose group of units is of rank one).

Let \( \mu(\epsilon) \) denote the finite cyclic group of complex roots of unity in \( Z[\epsilon] \). Hence, \( \{\pm 1\} \subseteq \mu(\epsilon) \). Let \( 2m \geq 2 \) denote the order of \( \mu(\epsilon) \) and let
\[ \zeta_{2m} = \exp(i\pi/m) \] be a generator of this cyclic group \( \mu(\epsilon) = \langle \zeta_{2m} \rangle \). Let \( \eta \in \mathbb{Z}[\epsilon] \) denote a fundamental unit of the quartic order \( \mathbb{Z}[\epsilon] \), i.e. assume that the group of units of this order \( \mathbb{Z}[\epsilon] \) is equal to \( \{ \zeta^n \mid \zeta \in \mu(\epsilon), n \in \mathbb{Z} \} \).

(Notice that, in the case that \( \mu(\epsilon) \neq \{ \pm 1 \} \), \( \eta \) could be a real quadratic unit or a totally real quartic unit. However, if \( \mu(\epsilon) = \{ \pm 1 \} \), then \( \eta \) is also a totally imaginary quartic unit, and by changing \( \eta \) into \(-\eta\), \(1/\eta\) or \(-1/\eta\) if necessary, we can assume that its minimal monic polynomial \( \Pi_\eta(X) \) is also of type \( (T) \).)

Then, \( \epsilon = \zeta \eta^n \) with \( \zeta \in \mu(\epsilon) \) and \( n \in \mathbb{Z} \setminus \{0\} \) and, if \( \zeta \in \mu(\eta) \), hence in particular if \( \mu(\epsilon) = \{ \pm 1 \} \), then \( \mathbb{Z}[\epsilon] = \mathbb{Z}[\eta^n] = \mathbb{Z}[\eta] \) and \( d_\eta = d_\epsilon \).

We will explain why it is reasonable to conjecture the following:

**Conjecture 8 (Compare with Theorems 1 and 3).** Let \( \epsilon \) be a totally imaginary quartic algebraic unit whose minimal monic quartic polynomial \( \Pi_\epsilon(X) \) is of type \( (T) \). Let \( \eta \) denote a fundamental unit of the totally imaginary quartic order \( \mathbb{Z}[\epsilon] \). Then, we can choose \( \eta = \epsilon \), except in the following cases:

1. The infinite family of exceptions for which \( \epsilon \) is a root of \( \Pi_\epsilon(X) = X^4 - 2bX^3 + (b^2 + 2)X^2 - (2b - 1)X + 1, b \geq 3, \) in which cases \( \epsilon = -1/\eta^2 \) where \( \eta = \epsilon^3 - 2b\epsilon^2 + (b^2 + 1)\epsilon - (b - 1) \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_\eta(X) = X^4 - X^3 + bX^2 + 1 \) of type \( (T) \), \( \mathbb{Z}[\eta] = \mathbb{Z}[\epsilon] \) and \( d_\eta = d_\epsilon = 16b^3 - 4b^2 - 128b^2 + 144b + 229 \).

2. The following sporadic exceptions:

   (a) i. \( \Pi_\epsilon(X) = X^4 - 3X^3 + 2X^2 + 1 \), in which case \( \zeta_3 = \epsilon^3 - 2\epsilon^2 + \epsilon - 1 \in \mathbb{Z}[\epsilon] \) and \( \epsilon = -\zeta_3 \eta^2 \) where \( \eta = \epsilon^2 - \epsilon \in \mathbb{Z}[\epsilon] \).
   
   ii. \( \Pi_\epsilon(X) = X^4 - 3X^3 + 5X^2 - 3X + 1 \), in which case in which \( \zeta_3 = \epsilon^3 - 3\epsilon^2 + 4\epsilon - 2 \in \mathbb{Z}[\epsilon] \) and \( \epsilon = \zeta_3 \eta^2 \) where \( \eta = -\epsilon^3 + 2\epsilon^2 - 3\epsilon + 1 \in \mathbb{Z}[\epsilon] \).
   
   iii. \( \Pi_\epsilon(X) = X^4 - 5X^3 + 8X^2 - 4X + 1 \), in which case \( \zeta_3 = -\epsilon^3 + 4\epsilon^2 - 5\epsilon + 1 \in \mathbb{Z}[\epsilon] \) and \( \epsilon = \zeta_3 \eta^3 \) where \( \eta = -\epsilon^2 + 3\epsilon - 1 \in \mathbb{Z}[\epsilon] \).

   In these three cases, \( \eta \) is a root of \( \Pi_\eta(X) = X^4 - 2X^3 + 2X^2 - X + 1 \) of type \( (T) \), \( \zeta_3 = \eta^2 - \eta \in \mathbb{Z}[\eta] \), \( \mathbb{Z}[\eta] = \mathbb{Z}[\epsilon] \) and \( d_\eta = d_\epsilon = 117 \).

   (b) \( \Pi_\epsilon(X) = X^4 - 5X^3 + 9X^2 - 5X + 1 \), in which case \( \epsilon = -\eta^2 \) where \( \eta = -\epsilon^3 + 4\epsilon^2 - 6\epsilon + 2 \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_\eta(X) = X^4 - X^3 + 3X^2 - X + 1 \) of type \( (T) \), \( \mathbb{Z}[\eta] = \mathbb{Z}[\epsilon] \) and \( d_\eta = d_\epsilon = 189 \).

   (c) i. \( \Pi_\epsilon(X) = X^4 - X^3 + 2X^2 + 1 \), in which case \( \epsilon = \eta^2 \) where \( \eta = -\epsilon^3 + \epsilon^2 - \epsilon \in \mathbb{Z}[\epsilon] \).
   
   ii. \( \Pi_\epsilon(X) = X^4 - 3X^3 + 3X^2 - X + 1 \), in which case \( \epsilon = 1/\eta^3 \) where \( \eta = -\epsilon + 1 \in \mathbb{Z}[\epsilon] \).
iii. $\Pi_1(X) = X^4 - 4X^3 + 6X^2 - 3X + 1$, in which case $\epsilon = -1/\eta^4$ where $\eta = \epsilon^3 - 3\epsilon^2 + 3\epsilon \in \mathbb{Z}[\epsilon]$.

iv. $\Pi_1(X) = X^4 - 5X^3 + 5X^2 + 3X + 1$, in which case $\epsilon = -\eta^6$ where $\eta = \epsilon^2 - 2\epsilon - 1 \in \mathbb{Z}[\epsilon]$.

v. $\Pi_1(X) = X^4 - 7X^3 + 14X^2 - 6X + 1$, in which case $\epsilon = -1/\eta^4$ where $\eta = \epsilon^2 - 4\epsilon + 2 \in \mathbb{Z}[\epsilon]$.

In these five cases, $\eta$ is a root of $\Pi_1(X) = X^4 - X^3 + 1$ of type $(T)$, $\mathbb{Z}[\eta] = \mathbb{Z}[\epsilon]$ and $d_\eta = d_\epsilon = 229$.

(d) i. $\Pi_1(X) = X^4 - 2X^2 + 3X^2 - X + 1$, in which case $\epsilon = -1/\eta^2$ where $\eta = \epsilon^3 - 2\epsilon^2 + 2\epsilon \in \mathbb{Z}[\epsilon]$.

ii. $\Pi_1(X) = X^4 - 3X^3 + X^2 + 2X + 1$, in which case $\epsilon = 1/\eta^3$ where $\eta = \epsilon^2 - 2\epsilon \in \mathbb{Z}[\epsilon]$.

iii. $\Pi_1(X) = X^4 - 5X^3 + 7X^2 - 2X + 1$, in which case $\epsilon = -\eta^4$ where $\eta = \epsilon^2 - 2\epsilon \in \mathbb{Z}[\epsilon]$.

In these three cases, $\eta$ is a root of $\Pi_1(X) = X^4 - X^3 + X^2 + 1$ of type $(T)$, $\mathbb{Z}[\eta] = \mathbb{Z}[\epsilon]$ and $d_\eta = d_\epsilon = 257$.

(e) $\Pi_1(X) = X^4 - 4X^3 + 7X^2 - 4X + 1$, in which case $\zeta_4 = \epsilon^3 - 4\epsilon^2 + 6\epsilon - 2 \in \mathbb{Z}[\epsilon]$ and $\epsilon = \zeta_4^2 \eta^3$, where $\eta = -\epsilon^3 + 3\epsilon^2 - 4\epsilon + 1 \in \mathbb{Z}[\epsilon]$ is a root of $\Pi_1(X) = X^4 - 2X^3 + X^2 + 1$ of type $(T)$, $\mathbb{Z}[\eta] = \mathbb{Z}[\epsilon]$ and $d_\eta = d_\epsilon = 272$.

(f) $\Pi_1(X) = X^4 - 13X^3 + 43X^2 - 5X + 1$, in which case $\epsilon = -\eta^3$ where $\eta = -\epsilon^3 + 6\epsilon^2 + 3\epsilon \in \mathbb{Z}[\epsilon]$ is a root of $\Pi_1(X) = X^4 - 2X^3 + 4X^2 - X + 1$ of type $(T)$, $\mathbb{Z}[\eta] = \mathbb{Z}[\epsilon]$ and $d_\eta = d_\epsilon = 1229$.

We will partially prove this conjecture. First, we will prove that in all these fifteen cases the given $\eta$ is indeed a fundamental unit of the quartic order $\mathbb{Z}[\epsilon]$.

Conversely, we will determine a fundamental unit of the quartic order $\mathbb{Z}[\epsilon]$ (i) in the case that $\mu(\epsilon) \neq \{\pm 1\}$, and (ii) in the case that the totally imaginary quartic number field $K_\epsilon = \mathbb{Q}(\epsilon)$ contains an imaginary quadratic subfield, a special situation of this latter case being:

**Theorem 9.** Let $\epsilon$ be a quartic unit whose minimal monic polynomial is of the form $\Pi_1(X) = X^4 - aX^3 + bX^2 - aX + 1 \in \mathbb{Z}[X]$, with $a \geq 0$ (if not, consider $-\epsilon$). Then, $\epsilon$ is totally imaginary and is not a complex root of unity if and only if $b \geq 3$ and $0 \leq a \leq \sqrt{4b - 11}$. In that case, $\mathbb{Q}(\epsilon)$ contains the imaginary quadratic field $\mathbb{Q}(\sqrt{-(4b - 8 - a^2)})$ and $d_\epsilon = ((b + 2)^2 - 4a^2)(4b - 8 - a^2)^2$. Let $\eta$ denote a fundamental unit of the quartic order $\mathbb{Z}[\epsilon]$.

By Theorem 10, we can choose $\eta = \epsilon$, except in 3 cases:

1. $\Pi_1(X) = X^4 - 3X^3 + 5X^2 - 3X + 1$, in which case $\epsilon = \eta^2$ where $\eta = -\epsilon^3 + 2\epsilon^2 - 2\epsilon \in \mathbb{Z}[\epsilon]$ is a root of $\Pi_1(X) = X^4 - X^3 - X^2 + X + 1$ of type $(T)$, $\mathbb{Z}[\eta] = \mathbb{Z}[\epsilon]$ and $d_\eta = d_\epsilon = 117$. 

2. \( \Pi_3(X) = X^4 - 5X^3 + 9X^2 - 5X + 1 \), in which case \( \epsilon = -\eta^2 \) where \( \eta = -\epsilon^3 + 4\epsilon^2 - 6\epsilon + 2 \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_3(X) = X^4 - X^3 + 3X^2 - X + 1 \) of type (T), \( \mathbb{Z}[\eta] = \mathbb{Z}[\epsilon] \) and \( d_{\eta} = d_{\epsilon} = 189 \).

3. \( \Pi_4(X) = X^4 - 4X^3 + 7X^2 - 4X + 1 \), in which case \( \zeta_4 = \epsilon^3 - 4\epsilon^2 + 6\epsilon - 2 \in \mathbb{Z}[\epsilon] \) and \( \epsilon = \zeta_4^2 \eta^2 \), where \( \eta = -\epsilon^3 + 3\epsilon^2 - 4\epsilon + 1 \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_4(X) = X^4 - 2X^3 + X^2 + 1 \) of type (T), \( \mathbb{Z}[\eta] = \mathbb{Z}[\epsilon] \) and \( d_{\eta} = d_{\epsilon} = 272 \).

**Proof.** Here, \( \eta + 1/\epsilon \) and \( \bar{\eta} + 1/\epsilon \) are roots of \( P(X) = X^2 - aX + b - 2 \), with \( d_P = a^2 - 4b + 8 \). If \( \epsilon \) is totally imaginary and is not a complex root of unity, then \( |\epsilon| \neq 1 \) and \( \eta + 1/\epsilon \) and \( \bar{\eta} + 1/\epsilon \) are distinct, hence not real, and \( d_P < 0 \), i.e., \( d_P \leq -3 \), and \( \mathbb{Q}(\epsilon + 1/\epsilon) = \mathbb{Q}(\sqrt{-4b - 8 - a^2}) \) is an imaginary quadratic subfield of \( \mathbb{Q}(\epsilon) \). Conversely, if \( d_P < 0 \), then its roots \( \eta + 1/\epsilon \) and \( \bar{\eta} + 1/\epsilon \) are not real and distinct, hence \( \epsilon \) is totally imaginary and \( |\epsilon| \neq 1 \), i.e. \( \epsilon \) is not a complex root of unity. \( \Box \)

5. Partial proof of the conjecture

The aim of this section is to partially settle Conjecture 8:

**Theorem 10.** Let \( \epsilon \) be a totally imaginary quartic algebraic unit, with \( \Pi_4(X) \) of type (T). Assume either (i) that \( \mathbb{Z}[\epsilon] \) contains non trivial (i.e. \( \neq \pm 1 \)) complex roots of unity, or (ii) that \( \mathbb{Q}(\epsilon) \) contains an imaginary quadratic subfield. Let \( \eta \) denote a fundamental unit of the order \( \mathbb{Z}[\epsilon] \). Then, we can choose \( \eta = \epsilon \), except in the following five cases:

1. \( \Pi_3(X) = X^4 - 3X^3 + 2X^2 + 1 \), in which case \( \zeta_3 = \epsilon^3 - 2\epsilon^2 + \epsilon - 1 \in \mathbb{Z}[\epsilon] \) and \( \epsilon = -\eta^2 \), where \( \eta = -\epsilon^3 + \epsilon^2 + 1 \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_3(X) = X^4 - 2X^3 + 2X^2 - X + 1 \) of type (T), \( \mathbb{Z}[\eta] = \mathbb{Z}[\epsilon] \) and \( d_{\eta} = d_{\epsilon} = 117 \) (case 2a.i of Conjecture 8).

2. \( \Pi_3(X) = X^4 - 3X^3 + 5X^2 - 3X + 1 \), in which case \( \zeta_3 = -\epsilon^3 + 3\epsilon^2 - 4\epsilon + 1 \in \mathbb{Z}[\epsilon] \) and \( \epsilon = \eta^3 \), where \( \eta = -\epsilon^3 + 3\epsilon^2 - 2\epsilon + 1 \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_3(X) = X^4 - 2X^3 + 2X^2 + X + 1 \) of type (T), \( \mathbb{Z}[\eta] = \mathbb{Z}[\epsilon] \) and \( d_{\eta} = d_{\epsilon} = 117 \) (case 2a.ii of Conjecture 8).

3. \( \Pi_3(X) = X^4 - 5X^3 + 8X^2 - 4X + 1 \), in which case \( \zeta_3 = -\epsilon^3 + 4\epsilon^2 - 5\epsilon + 1 \in \mathbb{Z}[\epsilon] \) and \( \epsilon = \zeta_3^3 \eta^3 \), where \( \eta = -\epsilon^3 + 3\epsilon - 1 \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_3(X) = X^4 - 2X^3 + 2X^2 - X + 1 \) of type (T), \( \mathbb{Z}[\eta] = \mathbb{Z}[\epsilon] \) and \( d_{\eta} = d_{\epsilon} = 117 \) (case 2a.iii of Conjecture 8).

4. \( \Pi_3(X) = X^4 - 5X^3 + 9X^2 - 5X + 1 \), in which case \( \zeta_3 = -\epsilon^3 + 5\epsilon^2 - 8\epsilon + 2 \in \mathbb{Z}[\epsilon] \) and \( \epsilon = -\eta^2 \), where \( \eta = -\epsilon^3 + 4\epsilon^2 - 6\epsilon + 2 \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_3(X) = X^4 - X^3 + 3X^2 - X + 1 \) of type (T), \( \mathbb{Z}[\eta] = \mathbb{Z}[\epsilon] \) and \( d_{\eta} = d_{\epsilon} = 189 \) (case 2b of Conjecture 8).
5. \( \Pi(X) = X^4 - 4X^3 + 7X^2 - 4X + 1 \), in which case \( \zeta_4 = e^3 - 4e^2 + 6e - 2 \in \mathbb{Z}[e] \) and \( e = \zeta_4^2 \eta^2 \), where \( \eta = -e^3 + 3e^2 - 4e + 1 \in \mathbb{Z}[e] \) is a root of \( \Pi(X) = X^4 - 2X^3 + X^2 + 1 \) of type \( (T) \), \( \mathbb{Z}[\eta] = \mathbb{Z}[e] \) and \( d_e = d_\eta = 272 \) (case 2e of Conjecture 8).

Let \( \zeta_{2m} \in \mathbb{Z}[e] \) be a generator of order \( 2m \geq 2 \) of the finite cyclic group \( \mu(e) = \langle \zeta_{2m} \rangle \) of the complex roots of unity in \( \mathbb{Z}[e] \). Then, \( \phi(2m) \) divides 4. Hence, \( \# \mu(e) = 2m \in \{2, 4, 6, 8, 10, 12\} \), and there are three cases:

1. If \( 2m \in \{8, 10, 12\} \), then \( \mathbb{Q}(e) \) contains no imaginary quadratic subfield \( k \) for which \( \mu(e) \subseteq k \). This case is dealt with in Section 5.1.

2. If \( 2m \in \{4, 6\} \), then \( \mathbb{Q}(e) \) contains an imaginary quadratic subfield \( k \) for which \( \mu(e) \subseteq k \), which may also happen in the case that \( 2m = 2 \). These cases are dealt with in Section 5.2, the key point being Proposition 12.

3. \( \mu(e) = \{\pm 1\} \) and \( \mathbb{Q}(e) \) contains no imaginary quadratic subfield.

5.1 The case that \( \mu(e) \) is of order 8, 10 or 12

In that case, \( \mathbb{Q}(e) = \mathbb{Q}(\zeta_{2m}) \) (for \( \mathbb{Q}(\zeta_{2m}) \subseteq \mathbb{Q}(e) \) and both fields have degree 4), hence \( \mathbb{Z}[\zeta_{2m}] \subseteq \mathbb{Z}[e] \subseteq \mathbb{Z}[\zeta_{2m}] \) (for \( e \) an algebraic integer and \( \mathbb{Z}[\zeta_{2m}] \) is the ring of algebraic integers of \( \mathbb{Q}(\zeta_{2m}) = \mathbb{Q}(e) \) and \( \mathbb{Z}[\zeta_{2m}] = \mathbb{Z}[e] \), which implies \( d_e = d_{\zeta_{2m}} \in \{256, 125, 144\} \).

**Proposition 11.** Assume that \( 2m \in \{8, 10, 12\} \) and that \( \mathbb{Z}[\zeta_{2m}] = \mathbb{Z}[e] \), i.e. that the ring of algebraic integers \( \mathbb{Z}[\zeta_{2m}] \) of the cyclotomic number field \( \mathbb{Q}(\zeta_{2m}) \) is generated by a totally imaginary quartic unit \( e \) with \( |e| > 1 \). Then, either (i) \( 2m = 10, e = \pm \zeta_8^a + \sqrt[4]{2} \) with \( a \in \{1, 2, 3, 4\} \) and \( e \) is a fundamental unit of \( \mathbb{Z}[e] = \mathbb{Z}[e_0] \) or (ii) \( 2m = 12, e = \zeta_8^a (1 + \zeta_8)/(2 + \sqrt{3}) \) with \( a \in \{1, 2, 4, 5, 7, 8, 10, 11\} \) and \( e \) is a fundamental unit of \( \mathbb{Z}[e] = \mathbb{Z}[e_{12}] \).

**Proof.** The result follows from [3, Théorème 8] or [5, Theorem 1.1], [3, Théorème 7] or [4, Theorem 5.1], and [3, Théorème 9]. We give a proof which does not use T. Nagell and L. Robertson’s results.

If \( 2m = 8 \), then \( \eta = 1 + \sqrt{2} \) is a fundamental unit of \( \mathbb{Z}[\zeta_8] \), we must have \( e = \zeta_8^a \eta^b \) with \( b \geq 1 \) and \( a \in \{1, 2, 3, 5, 6, 7\} \) (for \( |e| > 1 \) and \( e \) is not quadratic over \( \mathbb{Q} \)), and

\[
\begin{align*}
d_e &= 16 \sin^2(\pi a/4) \sin^2(3\pi a/4) \eta^{6b} \\
&\quad \times |1 - (-1)^b \zeta_8^{2a} / \eta^{2b}|^4 |1 - (-1)^b \zeta_8^{4a} / \eta^{2b}|^4.
\end{align*}
\]

Hence, \( d_e \to \infty \) as \( b \to \infty \), and it easily follows that \( d_e \) is never equal to 256 for \( b \geq 1 \).
If $2m = 10$, then $\eta = (1 + \sqrt{5})/2 > 1$ is a fundamental unit of $\mathbb{Z} [\zeta_5]$, we must have $\epsilon = \pm \zeta_5^a \eta^b$ with $b \geq 1$ and $a \in \{1, 2, 3, 4\}$ (for $|\epsilon| > 1$ and $\epsilon$ is not quadratic over $\mathbb{Q}$), and

$$d_\epsilon = 5\eta^{8b} |1 - (-1)^b \zeta_5^a / \eta^{2b}|^4 \left|1 - (-1)^b \zeta_5^{-a} / \eta^{-2b}\right|^4.$$ 

Hence, $d_\epsilon \to \infty$ as $b \to \infty$, and it easily follows that $d_\epsilon = 125$ if and only if $b = 1$ and $a \in \{1, 2, 3, 4\}$.

If $2m = 12$, then $\frac{1+3\sqrt{3}}{2} \eta$ is a fundamental unit of $\mathbb{Z} [\zeta_{12}]$, where $\eta = 1 + \sqrt{3}$, we must have $\epsilon = \zeta_{12}^a \left(\frac{1+3\sqrt{3}}{2} \eta\right)^b$ with $b \geq 1$, and

$$d_\epsilon = \sin^2 \left(\frac{\pi (2a + 9b)}{12}\right) \sin^2 \left(\frac{\pi (10a + 9b)}{12}\right) \eta^{8b}$$

$$\times \left|1 - \zeta_{12}^{-a+3b} \left(\frac{2}{\eta}\right)^b\right|^4 \left|1 - \zeta_{12}^{-a+3b} \left(\frac{2}{\eta}\right)^b\right|^4.$$ 

Hence, assuming that $a$ and $b \geq 1$ range over rational integers for which this discriminant is not equal to zero, $d_\epsilon \to \infty$ as $b \to \infty$, and it easily follows that $d_\epsilon = 144$ if and only if $b = 1$ and $a \in \{1, 2, 4, 5, 7, 8, 10, 11\}$. \hfill \qed

5.2 The case that $\mathbb{Q}(\epsilon)$ contains an imaginary quadratic field

If the quartic number field $K_\epsilon = \mathbb{Q}(\epsilon)$ contains an imaginary quadratic subfield $k$, then $d_k^2$ divides $d_{K_\epsilon}$, hence $d_k^2$ divides $d_\epsilon = (A_{K_\epsilon} : \mathbb{Z}[\epsilon])^2 d_{K_\epsilon}$, where $A_{K_\epsilon}$ is the ring of algebraic integers of $K_\epsilon$.

**Proposition 12.** Let $\epsilon$ be a totally imaginary quartic unit, with $|\epsilon| > 1$. Assume that $K_\epsilon = \mathbb{Q}(\epsilon)$ contains an imaginary quadratic subfield $k$ and that $\mu(\epsilon) \subseteq k$. Assume that $\epsilon$ is not a fundamental unit of the order $\mathbb{Z}[\epsilon]$. Let $\eta$ be a fundamental unit of this order. Write $\Pi_{\eta}(X) = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X]$ and $\epsilon = \zeta \eta^n$, with $\zeta \in \mu(\epsilon)$. Then, $2 \leq |n| \leq 5$ and $-1 \leq b \leq 7$.

**Proof.** Let $\epsilon'$ be the conjugate of $\epsilon$ over $k$ and $A_k = \mathbb{Z}[\omega]$ be the ring of algebraic integers of $k$ Then, $N_{K_\epsilon/k}(\epsilon) = \epsilon \epsilon'$ is an algebraic unit of $k$, hence a complex root of unity. Therefore, $|\epsilon'| = 1/|\epsilon| < 1$. We may assume that $|\eta| > 1$. Hence, $n \geq 2$. Notice that $\eta$ and $\eta^n$ are quadratic over $k$, otherwise $\eta^n$ would be in $k$, hence $\eta$ would be a complex root of unity and so $\epsilon = \zeta \eta^n$. Then, $A_k[\eta] = A_k[\eta^n]$ (for $\eta \in \mathbb{Z}[\epsilon]$ and $\zeta \in A_k$ yield $A_k[\eta^n] \subseteq A_k[\eta] \subseteq A_k[\epsilon] = A_k[\zeta \eta^n] = A_k[\eta^n]$). If $\eta^n = \alpha \eta - \beta$ with $\alpha, \beta \in A_k$, then $1 = (A_k[\eta] : A_k[\eta^n]) = |\alpha|^2 = |\eta^n - \eta^n|^2$.
|η − η′|^2. Indeed, \{1, ω, η, ωη\} and \{1, ω, η^n, ωη^n\} being two \(\mathbb{Z}\)-basis of the free \(\mathbb{Z}\)-modules \(A_k[η]\) and \(A_k[η^n]\), in writing \(α = a + bω\) and \(β = c + dω\), we have

\[
(A_k[η] : A_k[η^n]) = \begin{vmatrix}
1 & 0 & -c & dN_k/Q(ω) \\
0 & 1 & -d & -c - dTr_k/Q(ω) \\
0 & 0 & a & -bN_k/Q(ω) \\
0 & 0 & b & a + bTr_k/Q(ω)
\end{vmatrix}
= N_k/Q(a + bω).
\]

Hence,

\[|ε - ε'|^2 = |η^n - η''|^2 = |η - η'|^2 \quad (5)\]

and

\[|η|^n - 1/|η| ≤ |η^n - η''| = |η - η'| ≤ |η| + 1/|η| \quad (6)\]

As in the proof of Theorem 1, using (6), we obtain \(|η| ≤ (1 + \sqrt{5})/2\). Hence, \(|b| ≤ |η|^2 + 1/|η|^2 + 4 ≤ 7\) (express \(b\) in terms of the complex roots of \(Π_η(X)\)). Using (6) and Lemma 13, we obtain \(n ≤ 5\).

\[\text{Lemma 13. Let } η \text{ be a totally imaginary quartic unit. Assume that } |η| > 1. \text{ Then, } |η| ≥ |η_0| = 1.18375 \ldots, \text{ where } Π_η(X) = X^4 - X^3 + 1.\]

Proof. If \(\prod_{i=1}^4 (X - x_i) = X^4 - σ_1X^3 + σ_2X^2 - σ_3X + σ_4\), then \(\prod_{1 ≤ i < j ≤ 4} (X - x_i)x_j) = X^6 - σ_2X^5 + (σ_1σ_3 - σ_4)X^4 - (σ_1σ_3 - 2σ_2σ_4 + σ_2^2)X^3 + (σ_1σ_3σ_4 - σ_4^2)X^2 - σ_2σ_4^2X + σ_4^3\). In our situation, \(x_1 = η, x_2 = η, x_3 = η'\) and \(x_4 = η''\) are the complex roots of \(Π_η(X) = X^4 - aX^3 + bX^2 - cX + 1 ∈ \mathbb{Z}[X]\). Hence, \(|η|^2 > 1\) and \(1/|η|^2 < 1\) are real roots of

\[Q_η(X) = X^6 - bX^5 + (ac - 1)X^4 - (a^2 - 2b + c^2)X^3 + (ac - 1)X^2 - bX + 1.\]

The absolute values of its four other complex roots \(ηη', ηη'', ηη'\) and \(ηη''\) are equal to 1.

If \(c ≠ a\), then \(Q_η(0) = 1 > 0\) and \(Q_η(1) = -(a - c)^2 < 0\). Hence, using the dichotomy method, it is easy to compute numerical approximations to the only real root \(1/|η|^2\) in \([0, 1]\) of \(Q_η(X)\).

If \(c = a\), then \(Q_η(X) = (X - 1)^2R_η(X)\) with

\[R_η(X) = X^4 - (b - 2)X^3 + (a^2 - 2b + 2)X^2 - (b - 2)X + 1.\]
We may assume that \( \eta' = 1/\eta \) (for \( 1/\eta \) is also a root of \( \Pi_\eta(X) = X^4\Pi_\eta(1/X) \)). Therefore, \( |\eta|^2 > 1, 1/|\eta|^2 < 1, \eta/\bar{\eta} \) and \( \bar{\eta}/\eta \) are the four roots of \( R_\eta(X) \). Hence, \( |\eta|^2 + 1/|\eta|^2 > 2 \) and \( \eta/\bar{\eta} + \bar{\eta}/\eta \in [-2, 2] \) are roots of \( S(X) = X^2 - (b - 2)X + a^2 - 2b \). Hence, we must have \( d_S = (b + 2)^2 - 4a^2 > 0, S(2) < 0 \Rightarrow a^2 - 4b + 8 < 0, |\eta|^2 + 1/|\eta|^2 = (b - 2 + \sqrt{(b + 2)^2 - 4a^2})/2 \) and

\[
|\eta|^2 = \left( b - 2 + \sqrt{(b + 2)^2 - 4a^2} \right) + \sqrt{2b^2 - 4a^2 - 8 + 2(b - 2)\sqrt{(b + 2)^2 - 4a^2}})/4.
\]

Finally, we adapt the proof of [1, Lemma 2]: if \( 1 < |\eta| \leq 1.2 \), then \( -1 \leq b \leq 0 \). Using (4), we make the list of all the possible monic quartic polynomials \( \Pi_\eta(X) = X^4 - aX^3 + bX^2 - cX + 1 \) of type (T) in this range. For each \( \Pi_\eta(X) \) in this list, we compute \( Q_\eta(X) \) and numerical approximations to \( 1/|\eta|^2 \) if \( c \neq a \), whereas we use our formula above for \( |\eta|^2 \) in the case that \( c = a \). We obtain the desired result: if \( 1 < |\eta| \leq 1.2 \), then \( |\eta| \geq |\eta_0| \).

5.2.1 The case that \( \mu(\epsilon) \) is of order 4

**Lemma 14.** Let \( \epsilon \) be a totally imaginary quartic unit whose minimal monic polynomial \( \Pi_\epsilon(X) \) is of type (T). Assume that \( \mu(\epsilon) = \{ \pm 1, \pm \zeta_4 \} \). If \( \epsilon = \epsilon_0 \eta^2 \) for some \( \alpha \in \{ 0, 1, 2, 3 \} \) and some unit \( \eta \in \mathbb{Z}[\epsilon] \), then \( \Pi_\epsilon(X) = X^4 - 4X^3 + 7X^2 - 4X + 1, d_\alpha = 272, \zeta_4 = \epsilon^3 - 4\epsilon^2 - 6\epsilon - 2 \in \mathbb{Z}[\epsilon], \epsilon = \epsilon_0 \eta^2 \) and \( \eta = -\epsilon^3 + 3\epsilon^2 - 4\epsilon + 1 \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_\epsilon(X) = X^4 - 2X^3 + X^2 + 1 \) of type (T), \( \epsilon = -\eta^3 + \eta^2 + 1 \in \mathbb{Z}[\eta], \mathbb{Z}[\eta] = \mathbb{Z}[\epsilon], d_\eta = d_\epsilon = 272 \) and \( \mu(\epsilon) = \{ \pm 1, \pm \zeta_4 \} \) (case 5 of Theorem 10).

**Proof.** We may assume that \( |\epsilon| > 1 \). First, \( \eta \) is quadratic over \( k = \mathbb{Q}(\zeta_4) \) (if \( \eta \in \mathbb{Q}(\zeta_4) \), then \( \eta \) is a complex root of unity, and so is \( \epsilon = \zeta_4 \eta^2 \), a contradiction), and \( \eta^2 - \alpha \eta + \beta = 0 \) for some \( \alpha \) and \( \beta \) in \( \mathbb{Z}[\zeta_4] \). Then \( \beta \) is a root of unity in \( k \). Moreover, \( |\alpha|^2 = |\eta + \eta'|^2 = 1 \), by (5), and \( \alpha \) is also a complex root of unity in \( k \). Moreover, since \( \eta' = \pm \alpha \eta \) satisfies \( \eta'^2 - \alpha^2 \eta' + \alpha^2 \beta = 0 \), we may assume that \( \eta^2 - \eta + \beta = 0 \) with \( \beta \in \{ \pm 1, \pm \zeta_4 \} \).

We cannot have \( \beta = +1 \), otherwise \( \eta \) is a complex root of unity, and so is \( \epsilon \), a contradiction. By changing \( \eta \) and \( \epsilon \) into \( \bar{\eta} \) and \( \bar{\epsilon} \), we may assume that \( \beta \in \{ -1, -\zeta_4 \} \). If \( \beta = -1 \), then \( \eta = (1 + \sqrt{5})/2, \epsilon = \zeta_4 \eta^2 \) with \( a \in \{ 1, 3 \} \) (for \( a \) is quartic), hence \( \eta = -(\epsilon^2 + 2)/3 \notin \mathbb{Z}[\epsilon] \), a contradiction. Therefore, \( \beta = -\zeta_4, \Pi_\epsilon(X) = (X^2 - X + \beta)(X^2 - X + \bar{\beta}) = X^4 - 2X^3 + X^2 + 1 \) and \( \zeta_4 = \eta^2 - \eta \in \mathbb{Z}[\eta] \subseteq \mathbb{Z}[\epsilon] \), which implies \( \mathbb{Z}[\epsilon] = \mathbb{Z}[\zeta_4][\epsilon]/\mathbb{Z}[\zeta_4][\zeta_4^2 \eta^2]/\mathbb{Z}[\zeta_4][\eta^2] \subseteq \mathbb{Z}[\epsilon] \). Hence \( \mathbb{Z}[\epsilon] = \mathbb{Z}[\eta] \) and \( d_\epsilon = d_\eta \).

However, we have the following Table:
Since $\Pi(X)$ must be of type (T), we have $\epsilon = \zeta^3\eta^2 = (\eta^2 - \eta)^3\eta^2 = -\eta^3 + \eta^2 + 1$ and $\Pi(X) = X^4 - 4X^3 + 7X^2 - 4X + 1$. Finally, using $\Pi_1(X)$, we compute $(q_{i,j})_{1 \leq i,j \leq 4} = (p_{i,j})_{1 \leq i,j \leq 4}^{-1}$, where $\epsilon^{-1} = \sum_{i=1}^{4} p_{i,2} \eta^{-i-1}$, to obtain $\eta = \sum_{i=1}^{4} q_{i,2} \epsilon^{i-1}$. Then, using $\Pi_{2}(X)$, we compute $\zeta = \eta^3 - \eta$ in $\mathbb{Z}[\epsilon]$, which completes the proof (since $d_\eta = 272 \not\in \{256, 125, 144\}$, the last assertion follows from the argument just before Proposition 11).

5.2.2 The case that $\mu(\epsilon)$ is of order 6

**Lemma 15.** Let $\epsilon$ be a totally imaginary quartic unit whose minimal monic polynomial $\Pi(X)$ is of type (T). Assume that $\mu(\epsilon) = \{\pm 1, \pm \zeta_3, \pm \zeta_3^2\}$. If $\epsilon = \pm \zeta_3^3\eta^3$ for some $\alpha \in \{0, 1, 2\}$ and some unit $\eta \in \mathbb{Z}[\epsilon]$, then $\Pi(X) = X^4 - 4X^3 + 8X^2 - 4X + 1$, $d_\epsilon = 117$, $\zeta_3 = -\epsilon^3 + 4\epsilon^2 - 5\epsilon + 1 \in \mathbb{Z}[\epsilon]$, $\epsilon = \zeta_3^3\eta^3$ and $\eta = -\epsilon^2 + 3\epsilon - 1 \in \mathbb{Z}[\epsilon]$ is a root of $\Pi_0(X) = X^4 - 2X^3 + 2X^2 - X + 1$ of type (T), $\epsilon = -\eta^3 + \eta^2 + 1 \in \mathbb{Z}[\eta]$, $\mathbb{Z}[\eta] = \mathbb{Z}[\epsilon]$, $d_\eta = d_\epsilon = 117$ and $\mu(\epsilon) = \{\pm 1, \pm \zeta_3, \pm \zeta_3^2\}$ (case 3 of Theorem 10).

**Proof.** We may assume that $|\epsilon| > 1$. Again, $\eta^2 - \alpha \eta + \beta = 0$ for some $\alpha \in \mathbb{Z}[\zeta_3]$ and $\beta$ a sixth root of unity. Since $|(\eta + \eta')^2 - \eta \eta'|^2 = |\alpha^2 - \beta|^2 = 1$, by (5), we have $|\alpha|^2 \leq 2$. Hence, either $\alpha = 0$, which would imply $|\eta|^2 = |1 - \beta^2| = 1$, a contradiction, or $\alpha \in \{\pm 1, \pm \zeta_3, \pm \zeta_3^2\}$. Since $\eta^3 = \pm \alpha^2 \eta$ satisfies $\eta^3 - \pm \alpha^2 \eta^2 = 0$, we may assume that $\eta^2 - \eta + \beta = 0$ with $\beta \in \{\pm 1, \pm \zeta_3, \pm \zeta_3^2\}$. Here again, we cannot have $\beta = -1$. If $\beta = -1$, then $\eta = (1 + \sqrt{5})/2, \epsilon = \zeta_3^2 \eta^3$ with $\alpha \in \{1, 2\}$, hence $\eta = (\pm 3 - 5)/8 \notin \mathbb{Z}[\epsilon]$, a contradiction. Hence, by changing $\eta$ and $\epsilon$ into $\bar{\eta}$ and $\bar{\epsilon}$, we may assume that $\beta \in \{\pm \zeta_3\}$. Hence, $\zeta_3 \in \mathbb{Z}[\eta]$, and here again $\mathbb{Z}[\epsilon] = \mathbb{Z}[\eta]$ and $d_\epsilon = d_\eta$.

We have $\Pi_0(X) = X^4 - 2X^3 + (1 + \beta + \bar{\beta})X^2 - (\beta + \bar{\beta})X + 1$. If $\beta = \zeta_3$, then $|(\eta + \eta')^2 - \eta \eta'|^2 = |1 - \zeta_3|^2 = 3$, which contradicts (5). Hence, $\beta = \zeta_3, \zeta_3 = \eta^2 - \eta, \Pi_{\pm}(X) = X^4 - 2X^3 + 2X^2 - X + 1, d_\eta = 117$ and we have the following Table:

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\Pi_{\pm \zeta_3 \eta^2}(X)$</th>
<th>$d_{\pm \zeta_3 \eta^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$X^4 \pm X^3 + 5X^2 \mp X + 1$</td>
<td>9477 = 92 · 117</td>
</tr>
<tr>
<td>1</td>
<td>$X^4 \pm 4X^3 - 8X^2 \pm 5X + 1$</td>
<td>117</td>
</tr>
<tr>
<td>2</td>
<td>$X^4 \mp 5X^3 + 8X^2 \mp 4X + 1$</td>
<td>117</td>
</tr>
</tbody>
</table>
Since \( \Pi_c(X) \) must be of type (T), we have \( \epsilon = \left( \frac{3}{2} \right) \eta^3 = (\eta^2 - \eta) \eta^3 = -\eta^3 + \eta^2 + 1 \) and \( \Pi_c(X) = X^4 - 5X^3 + 8X^2 - 4X + 1 \). □

5.2.3 The case that \( \pm \epsilon \) is a square in \( \mathbb{Z}[\epsilon] \)

Let \( \epsilon \) be a totally imaginary quartic algebraic unit. We may assume that \( \Pi_c(X) \) is of type (T), by changing \( \epsilon \) into \( -\epsilon \), \( 1/\epsilon \) or \( -1/\epsilon \), if necessary. Let \( \eta \) be a fundamental unit of the order \( \mathbb{Z}[\epsilon] \). Assume that \( \epsilon = \pm \eta^n \) for some \( n \in \mathbb{Z} \setminus \{0\} \). Then, \( \eta \) is a totally imaginary quartic unit, \( \mathbb{Z}[\epsilon] = \mathbb{Z}[\eta] \), and \( d_\epsilon = d_\eta \).

We may assume that \( \Pi_\eta(X) \) is of type (T), by changing \( \eta \) into \( -\eta \), \( 1/\eta \) or \( -1/\eta \), if necessary.

**Lemma 16 (Compare with Lemma 5).** Let \( \epsilon \) be a totally imaginary quartic algebraic unit, with \( \Pi_c(X) \) of type (T). Then, \( \pm \epsilon \) is a square in \( \mathbb{Z}[\epsilon] \) if and only we are in one of the seven following cases:

1. \( \Pi_c(X) = X^4 - 2bX^3 + (b^2 + 2)X^2 - (2b - 1)X + 1, b \geq 1, \) in which cases \( \epsilon = -1/\eta^2 \) where \( \eta = \epsilon^3 - 2b\epsilon^2 + (b^2 + 1)\epsilon - (b - 1) \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_\eta(X) = X^4 - X^3 + bX^2 + 1 \) of type (T), and \( d_\epsilon = d_\eta = 16b^4 - 4b^3 - 12b^2 + 114b + 229 \) (cases 1, 2c.iii and 2d.i of Conjecture 8).

2. \( \Pi_c(X) = X^4 - X^3 + 2X^2 + 1, \) in which case \( \epsilon = \eta^2 \) where \( \eta = -\epsilon^3 + \epsilon^2 - \epsilon \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_\eta(X) = X^4 - X^3 + 1 \) of type (T), and \( d_\epsilon = d_\eta = 229 \) (case 2c.i of Conjecture 8).

3. \( \Pi_c(X) = X^4 - 3X^3 + 2X^2 + 1, \) in which case \( \epsilon = -\eta^{-2} \) where \( \eta = -\epsilon^2 + \epsilon + 1 \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_\eta(X) = X^4 - 2X^3 + 2X^2 - X + 1 \) of type (T), and \( d_\epsilon = d_\eta = 117 \) (case 1 of Theorem 10).

4. \( \Pi_c(X) = X^4 - 3X^3 + 5X^2 - 3X + 1, \) in which case \( \epsilon = \eta^2 \) where \( \eta = -\epsilon^3 + 2\epsilon^2 - 2\epsilon \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_\eta(X) = X^4 - X^3 - X^2 + X + 1 \) of type (T), and \( d_\epsilon = d_\eta = 117 \) (case 2 of theorem 10).

5. \( \Pi_c(X) = X^4 - 5X^3 + 9X^2 - 5X + 1, \) in which case \( \epsilon = -\eta^2 \) where \( \eta = -\epsilon^3 + 4\epsilon^2 - 6\epsilon + 2 \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_\eta(X) = X^4 - X^3 + 3X^2 - X + 1 \) of type (T), and \( d_\epsilon = d_\eta = 189 \) (case 4 of Theorem 10).

6. \( \Pi_c(X) = X^4 - 5X^3 + 5X^2 + 3X + 1, \) in which case \( \epsilon = -\eta^{-2} \) where \( \eta = -\epsilon^2 + 2\epsilon + 2 \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_\eta(X) = X^4 - 3X^3 + 3X^2 - X + 1 \) of type (T), and \( d_\epsilon = d_\eta = 229 \) (case 2c.iv of Conjecture 8).

7. \( \Pi_c(X) = X^4 - 5X^3 + 7X^2 - 2X + 1, \) in which case \( \epsilon = -\eta^{-2} \) where \( \eta = \epsilon^3 - 4\epsilon^2 + 4\epsilon \in \mathbb{Z}[\epsilon] \) is a root of \( \Pi_\eta(X) = X^4 - 2X^3 + 3X^2 - X + 1 \) of type (T), and \( d_\epsilon = d_\eta = 257 \) (case 2d.iii of Conjecture 8).

**Proof.** Assume that \( \epsilon = \pm \eta^2 \) or \( \pm \eta^{-2} \) for some \( \eta \in \mathbb{Z}[\epsilon] \), with \( \Pi_\eta(X) = X^4 - aX^3 + bX^2 - cX + 1 \) of type (T). Hence, \( |\epsilon| \leq a \). The index \( (\mathbb{Z}[\eta] : \mathbb{Z}[\eta^2]) \) is equal to \( |a^2 + c^2 - abc| \). Hence, we must have \( |a^2 + c^2 - abc| = 1 \).

If \( c = 0 \), then \( 1 = |a^2 + c^2 - abc| = a^2 \), hence \( a = 1 \) and we are in the first or the second case.
Now, assume that $c \neq 0$. Then, $1 \leq |c| \leq a$. If $b \geq 9$, then $b \geq \sqrt{4b+24} + 1 \geq a + 1 \geq |c| + 1$ and we obtain $7 \leq b - 2 \leq |c| b - (1 + c^2) \leq a |b - (a^2 + c^2) \leq |abc - a^2 - c^2|$. Hence, $-1 \leq b \leq 8$ and $1 \leq |c| \leq a \leq \sqrt{4b+24}$. In this range, there are 14 triplets $(a, b, c)$ for which $|a^2 + c^2 - abc| = 1$. By getting rid of those for which $\Pi_{\eta}(X) = X^4 - aX^3 + bX^2 - cX + 1$ is not of type (T), we fall in one of the five remaining last cases. Finally, by choosing between the four units $\pm \eta^2$ or $\pm \eta^{-2}$ the ones whose minimal monic polynomials are of type (T), we complete the proof of this Lemma.

\[ \square \]

5.2.4 The case that \( \pm \varepsilon \) is an \( n \)th power in \( \mathbb{Z}[\varepsilon] \)

**Lemma 17.** Let \( \varepsilon \) be a totally imaginary quartic algebraic unit such that \( \varepsilon = \pm \eta^n \), where \( \eta \in \mathbb{Z}[\varepsilon] \) is therefore a totally imaginary quartic algebraic unit whose minimal monic polynomial \( \Pi_{\eta}(X) = X^4 - aX^3 + bX^2 - cX + 1 \) may be assumed to be of type (T). Then, \( \mathbb{Z}[\eta] = \mathbb{Z}[\eta^n] \) and \( d_{\eta^n} = d_{\eta} \). Moreover, in the range $-1 \leq b \leq 10000$, we have \( d_{\eta^n} = d_{\eta} \) for some \( n \in \{3, \ldots, 10\} \) if and only if we are in one of the following cases:

1. \( d_{\eta^3} = d_{\eta} \) if and only if we are in one of the following four cases:
   (a) \( \Pi_{\eta}(X) = X^4 - X^3 + 1 \), in which case \( \Pi_{1/\eta^3}(X) = X^4 - 3X^3 + 3X^2 - X + 1 \) is of type (T) and \( d_{\eta^3} = d_{\eta} = 229 \).
   (b) \( \Pi_{\eta}(X) = X^4 - X^3 + 2X^2 + 1 \), in which case \( \Pi_{-\eta}(X) = X^4 - 5X^3 + 5X^2 + 3X + 1 \) is of type (T) and \( d_{\eta^3} = d_{\eta} = 229 \).
   (c) \( \Pi_{\eta}(X) = X^4 - X^3 + X^2 + 1 \), in which case \( \Pi_{-1/\eta^3}(X) = X^4 - 3X^3 + X^2 + X + 1 \) is of type (T) and \( d_{\eta^3} = d_{\eta} = 257 \).
   (d) \( \Pi_{\eta}(X) = X^4 - 2X^3 + 4X^2 - X + 1 \), in which case \( \Pi_{-1/\eta}(X) = X^4 - 13X^3 + 43X^2 - 5X + 1 \) is of type (T) and \( d_{\eta^3} = d_{\eta} = 1229 \).

2. \( d_{\eta^4} = d_{\eta} \) if and only if we are in one of the following two cases:
   (a) \( \Pi_{\eta}(X) = X^4 - X^3 + 1 \), in which case \( \Pi_{-1/\eta^4}(X) = X^4 - 4X^3 + 6X^2 - 3X + 1 \) is of type (T) and \( d_{\eta^4} = d_{\eta} = 229 \).
   (b) \( \Pi_{\eta}(X) = X^4 - X^3 + X^2 + 1 \), in which case \( \Pi_{-\eta^4}(X) = X^4 - 5X^3 + 7X^2 - 2X + 1 \) is of type (T) and \( d_{\eta^4} = d_{\eta} = 257 \).

3. \( d_{\eta^6} = d_{\eta} \) if and only if \( \Pi_{\eta}(X) = X^4 - X^3 + 1 \), in which case \( \Pi_{-1/\eta^6}(X) = X^4 - 5X^3 + 5X^2 + 3X + 1 \) and \( d_{\eta^6} = d_{\eta} = 229 \).

4. \( d_{\eta^7} = d_{\eta} \) if and only if \( \Pi_{\eta}(X) = X^4 - X^3 + 1 \), in which case \( \Pi_{-\eta^7}(X) = X^4 - 7X^3 + 14X^2 - 6X + 1 \) and \( d_{\eta^7} = d_{\eta} = 229 \).

5. \( d_{\eta^8}, d_{\eta^9}, d_{\eta^9} \) and \( d_{\eta^{10}} \) are never equal to \( d_{\eta} \).

**Proof.** Noticing that \( \Pi_{\eta^n}(X^n) = \prod_{k=0}^{n-1} \Pi_{\eta}(\zeta_k X) \), we compute \( \Pi_{\eta^n}(X) \) and \( d_{\eta^n} \) from the knowledge of \( \Pi_{\eta}(X) \). For example, if \( n = 3 \), there are 6 triplets
propositions are obtained for the following polynomials: $P_1(X)$ and $P_2(X)$. Then, 0 ≤ d₂ < b, and $d_\eta = d_\epsilon$, except in the case that $P_2(X) = X^3 + 3X^2 + 2X + 1$ where $d_\epsilon = 257, \epsilon = \eta^{-3}$ with $\eta = \epsilon^2 - \epsilon \in \mathbb{Z}[\epsilon]$ and $\Pi_\eta(X) = X^4 - X^3 + X^2 + 1$.

**Proof.** Suppose that $b_\epsilon \leq b_\eta$. We have $|\eta|^6 + 1/|\eta|^6 - 4 \leq |\epsilon|^2 + 1/|\epsilon|^2 - 4 \leq b_\epsilon < b_\eta \leq |\eta|^2 + 1/|\eta|^2 + 4$. Hence, 0 < $|\log |\eta|| < 0.393$. By (4), there are 8 such possible quartic polynomials $\Pi_\eta(X)$ of type (T). They all satisfy −1 ≤ b ≤ 2. Hence, we would have −1 ≤ b ≤ 2. By (4), there are 22 quartic polynomials $P(X)$ of type (T) with −1 ≤ b ≤ 2, and the repeated discriminants are 117, 229 and 257 and are obtained for the following polynomials:

| b    | $P(X)$              | $d_\eta$ | $|\log(\epsilon_\eta)|$ |
|------|---------------------|----------|-------------------------|
| −1   | $X^4 - X^3 - X^2 + X + 1$ | 117       | 0.27176 ...              |
| 0    | $X^4 - X^3 + 1$      | 229       | 0.16868 ...              |
| 1    | $X^4 - X^3 + X^2 + 1$ | 257       | 0.22106 ...              |
| 1    | $X^4 - 3X^3 + X^2 + 2X + 1$ | 257       | 0.25632 ...              |
| 2    | $X^4 - 2X^3 + 2X^2 - X + 1$ | 117       | 0.27176 ...              |
| 2    | $X^4 - 3X^3 + 2X^2 + 1$ | 117       | 0.54353 ...              |
| 2    | $X^4 - X^3 + 2X^2 + 1$ | 229       | 0.33737 ...              |

This proves the Lemma. □

5.2.5 The case that $\Pi_2(X) = X^4 - X^3 + bX^2 + 1 \in \mathbb{Z}[X], b \geq 3$

**Proposition 18.** Let $\epsilon$ be a complex root of $\Pi_2(X) = X^4 - X^3 + bX^2 + 1 \in \mathbb{Z}[X], b \geq 3$. Then, $\epsilon$ is a totally imaginary quartic unit, $d_\epsilon = 16b^4 - 4b^3 - 128b^2 + 144b + 229 > 0$ is odd, $\mu(\epsilon) = \{\pm 1\}$, hence we may assume that $|\epsilon| > 1$, $\sqrt{b - 1} \leq |\epsilon| \leq \sqrt{b}$, $d_\epsilon$ is asymptotic to $16|\epsilon|^b$ as $b \to \infty$, and $\epsilon$ is a fundamental unit of the quartic order $\mathbb{Z}[\epsilon]$. It follows that $\epsilon = \epsilon^2$ which is a root of $\Pi_2(X) = X^4 + (2b - 1)X^3 + (b^2 + 2)X^2 + 2bX^2 + 1$ is such that $d_\epsilon = d_\eta$ is asymptotic to $16|\epsilon|^b$ as $b \to \infty$. Finally, $\mathbb{Q}(\epsilon)$ contains no imaginary quadratic subfield.

**Proof.** Since $X^4 - X^3 + 1$ has no real root, we have $\Pi_2(x) \geq x^4 - x^3 + 1 > 0$ for $x$ real, and $\epsilon$ is totally imaginary. Since $\Pi_2(x)$ is $\mathbb{Q}$-irreducible, $\epsilon$ is a
totally imaginary quartic unit. We now that \( \mu(\epsilon) \) is of order 8, 10, 12, in which cases \( d_x = 256, 125 \) or 144, respectively, or of order 4 or 6, in which cases \( 4^2 \) or \( 3^2 \) divides \( d_x \), respectively. Since \( d_x \) is odd, not divisible by 3 and never equal to 125, we have \( \mu(\epsilon) = \{ \pm 1 \} \). Since \( 1/|\epsilon|^2 \) is the only real root in \((0, 1)\) of \( Q(X) = X^6 - bX^5 - X^4 + (2b - 1)X^3 - X^2 - bX + 1 \) (see the proof of Lemma 13), and since \( Q(1/b) = (b^4 - b^3 - 2b^2 + 1)/b^6 > 0 \) and \( Q(1) = -1 < 0 \), we have \( |\epsilon|^2 < b \). Since \( Q(1/(b - 1)) < 0 \), we have \( |\epsilon|^2 > b - 1 \). Let \( \eta \) be fundamental unit of \( \mathbb{Z}[\epsilon] \). Suppose that \( \epsilon = \pm \eta^a \) with \( |n| \geq 2 \). We have \( |n| \geq 3 \), by Lemma 16. Hence, by (8) below, we have

\[
\begin{align*}
\epsilon &= \epsilon_n \leq 16(|\eta| + 1/|\eta|)^8 \leq 16(|\epsilon|^{1/3} + 1/|\epsilon|^{1/3})^8 \\
&\leq 16(b^{1/6} + 1/b^{1/6})^8,
\end{align*}
\]

which implies \( b < 5 \). Hence, \( \epsilon \) is indeed a fundamental unit of \( \mathbb{Z}[\epsilon] \) for \( b \geq 5 \). For \( b = 3 \), we have \( |\epsilon| = 1.67635 \ldots \) and for \( b = 4 \), we have \( |\epsilon| = 1.95375 \ldots \). By Lemma 13, if \( \epsilon = \pm \eta^a \) for some \( \eta \in \mathbb{Z}[\epsilon] \), then \( |n| \leq 4 \). We then use Proposition 17 to settle these two cases. Let us prove the last assertion. The cubic resolvent of \( X^4 \Pi_r(1/X) = X^4 + bX^2 - X + 1 \) is \( R(X) = X^3 - 2bX^2 + (b^2 - 4)X + 1 \), and \( d_{R(X)} = d_{\Pi_r(X)} = d_x > 0 \). It is easy to see that \( R(X) \) is \( \mathbb{Q} \)-irreducible and that \( d_{R(X)} \) is never a perfect square. It follows that \( \mathbb{Q}(\epsilon) \) is a non-normal quartic field and that the Galois group of its normal closure is the symmetric group \( S_4 \) (see [6, Theorem 7.5.4] or [7, Theorem 80]). In particular, \( \mathbb{Q}(\epsilon) \) contains no quadratic subfield. \( \square \)

In the same spirit, we have:

**Proposition 20.** Let \( \epsilon \) be a complex root of \( P(X) = X^4 - aX^3 + bX^2 - aX + 1 \in \mathbb{Z}[X] \) with \( b \geq 4 \) and \( 0 \leq a < \sqrt{4b - 11} \). Then, \( \epsilon \) is a totally imaginary quartic unit, it is not a complex root of unity, hence we may assume that \( |\epsilon| > 1 \), \( d_x = ((b + 2)^2 - 4a^2)(4b - a^2 - 8)^2 \), \( \sqrt{b - 4} \leq |\epsilon| \leq \sqrt{b} \) and

\[
5|\epsilon|^4 \leq d_x = d_P \leq 17|\epsilon|^8. \tag{7}
\]

Moreover, if \( b = (a^2 - 11)/4 \), then \( d_x \) is asymptotic to \( 9|e|^4 \) as \( a \to \infty \), whereas if \( a \) is fixed, then \( d_x \) is asymptotic to \( 16|\epsilon|^8 \) as \( b \to \infty \).

**Proof.** We use the explicit formula for \( |\epsilon|^2 \) obtained in the proof of Lemma 13. Since \( d_P \geq 9((b + 2)^2 - 4(4b - 11)) = 9(b^2 - 12b + 48) \) and

\[
4|\epsilon|^2 \leq b - 2 + \sqrt{(b + 2)^2} + \sqrt{2b^2 - 8 + 2(b - 2)} \sqrt{(b + 2)^2} \leq 4b,
\]

the lower bound in (7) holds true for \( b \geq 23 \), and we numerically checked it for \( 3 \leq b \leq 22 \). Now, we have \( d_P \leq (b + 2)^2(4b - 8)^2 = 16b^4 - 4b^2 \leq 16b^4 \). On the other hand, we have \( (b + 2)^2 - 4a^2 \geq (b + 2)^2 - 4(4b - 11) =
The fundamental unit of some quadratic, cubic or quartic orders

\[ b^2 - 12b + 48 \text{ and } 2b^2 - 4a^2 - 8 \geq 2b^2 - 4(4b - 11) - 8 = 2b^2 - 16b + 36. \]

Hence,

\[
4|\epsilon|^2 \geq b - 2 + \sqrt{b^2 - 12b + 48} + \sqrt{2b^2 - 16b + 36 + 2(b - 2)\sqrt{b^2 - 12b + 48}}
\]

and \(|\epsilon|^2 \geq b - 4. \) Since, \( d_P \leq 16b^4 \leq 17(b - 4)^4 \leq 17|\epsilon|^8 \) for \( b \geq 266, \) the upper bound in (7) holds true for \( b \geq 266, \) and we numerically checked it for \( 3 \leq b \leq 265. \)

5.3 Proof of Theorem 10

Assume that \( \epsilon \) is not a fundamental unit of \( \mathbb{Z}[\epsilon]. \) We prove that we are in one of the five considered cases. Then, \( \epsilon = \zeta \eta^m \) with \( \zeta \in \mu(\epsilon), \eta \in \mathbb{Z}[\epsilon] \) and \(|n| \geq 2. \) By Proposition 11, we may assume that \( K_\epsilon = \mathbb{Q}(\epsilon) \) contains an imaginary quadratic subfield \( k \) such that \( \mu(\epsilon) \subseteq k. \) In particular, \( \mu(\epsilon) = \{\pm 1\}, \mu(\epsilon) = \{\pm 1, \pm \zeta_4\} \) or \( \mu(\epsilon) = \{\pm 1, \pm \zeta_5, \pm \zeta_5^2\}. \) By Proposition 12, we have \(|n| \leq 5. \) By Lemmas 14 and 16, we may assume that \(|n| \in \{3, 5\}. \) By Lemmas 15, 17 and the last assertion of Proposition 19, we may assume that \(|n| = 5. \) Finally, Lemma 17 settles this case. Conversely, in all these five cases, \( \eta \) is the fundamental unit of \( \mathbb{Z}[\epsilon] \) (apply the same reasoning to \( \eta \) instead of applying it to \( \epsilon. \) In the same way, we can prove that in all the fifteen cases of Conjecture 8, the given \( \eta \) is indeed a fundamental unit of the quartic order \( \mathbb{Z}[\epsilon] \).

6. How to prove conjecture 8?

Theorem 10 and Lemmas 16, 17 and 18 show that if \( \epsilon \) is a totally imaginary algebraic quartic unit, with \( \Pi_\epsilon(X) = X^4 - a_\epsilon X^3 + b_\epsilon X^2 - c_\epsilon X + 1 \in \mathbb{Z}[X] \) of type (T), and if \( \epsilon \) is not a fundamental unit of the order \( \mathbb{Z}[\epsilon], \) then either we are in one of the cases of Conjecture 8, or \(|b_\epsilon| > 10000, \) or \( \pm \epsilon \) is at least an eleventh power of a unit \( \eta \in \mathbb{Z}[\epsilon] \) (and these ranges could be easily extended). We now explain which result whose proof eludes us would enable us to settle Conjecture 8.

Proposition 21. Assume \( P(X) \) is of type (T). Let \( \epsilon \) be any one of the two complex roots of \( P(X) \) satisfying \(|\epsilon| > 1. \) Then,

\[
0 < d_\epsilon \leq 16(|\epsilon| + 1/|\epsilon|)^8 \leq 4096|\epsilon|^8.
\]

Proof. We have \( d_P = 16 \Im(\epsilon)^2 \Im(\epsilon')^2 |\epsilon - \epsilon'|^4 |\epsilon - \bar{\epsilon}'|^4. \) □
This Proposition and Propositions 19 and 20 make it reasonable to conjecture the following result on quartic polynomials which would enable us to prove Conjecture 8:

**Conjecture 22.** There exists some explicit constant $C > 0$ such that for any totally complex quartic algebraic unit $\epsilon$ satisfying $|\epsilon| > 1$ we have $d_\epsilon \geq C|\epsilon|^4$. In particular, there would be only finitely many totally complex quartic algebraic units of bounded discriminant, and if $|\epsilon|$ were large enough then $\epsilon$ would be either a fundamental unit of the quartic order $\mathbb{Z}[\epsilon]$ or its square, up to a product by a complex root of unity in $\mathbb{Z}[\epsilon]$.

**References**


