RAMANUJAN MODULAR FORMS
AND THE KLEIN QUARTIC

GILLES LACHAUD

friendly dedicated to Mikhail Tsfasman

Abstract. In one of his notebooks, Ramanujan gave some algebraic relations between three theta functions of order 7. We describe the automorphic character of a vector valued mapping made from these theta series. This provides a systematic way to establish old and new identities on modular forms for the congruence subgroup of level 7, above all the parametrization of the Klein quartic. From an historic point of view, this shows that Ramanujan discovered the main properties of this curve with his own means. As an application, we introduce a \( L \)-series in four different ways, generating the number of points of the Klein quartic over finite fields. From this we deduce the structure of the Jacobian of a suitable form of the Klein quartic over finite fields, and some congruence properties on the number of its points.

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1. Introduction

In the complex projective space \( \mathbb{P}^2(\mathbb{C}) \), the *Klein quartic* \( X \) is the curve of genus 3 by the homogeneous equation

\[
\Phi_4(x, y, z) = 0,
\]

where

\[
\Phi_4(x, y, z) = x^3y + y^3z + z^3x.
\]

We plan firstly to discuss on a specific modular parametrization of the Klein quartic by some theta constants \( a, b, c \) of order 7. We show that the main properties of these series are deduced from the modularity of a vector valued holomorphic map (Theorem 3.2). These theta series are written in a set of six formulas appearing in

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Ramanujan’s Notebooks [28, bottom of p. 300]; this passage is reproduced here as Appendix A for convenience).

Felix Klein discovered in 1879 a modular parametrization of the eponym quartic [19], which reduces ultimately to an identity between the theta series \(a, b, c\), leading to an isomorphism from the modular curve \(X(7)\) to \(X\). The work of Klein is explained and cleared in the seminal report of Elkies [10]. This is precisely this identity which was obtained thirty years later by Ramanujan, who was most likely unaware of the results of Klein.

In spite of the fact that such a parametrization is essentially unique, it appears under many different forms, and we find worthwhile to offer a unified presentation of the functions involved in this construction. In particular, most of the “theta constant identities” of order 7 can be ultimately seen as equations defining dominant morphisms of the Klein quartic to its quotient curves. As we shall see, all the necessary tools for a basic geometrical study of the Klein quartic have been provided by Ramanujan, with his own means.

It would be interesting to describe in detail the history of these constructions; we shall content here with refering to the original contributions of Klein and Ramanujan, to the analysis of some of the aforementioned formulas of Ramanujan by Berndt & Zhang [3, 4] and Craig [6], and to the works of Farkas & Kra [11] and Liu [22], where the reader will find some of the formulas obtained here in another guise.

Actually, the present work is related to a question of Mikhail Tsfasman: can one express the modular form of level 49 and of weight 2 in terms of Dedekind’s eta function? A tentative answer is given by the identity of Prop. 8.2. This identity, and some other appearing here, and involving Eisenstein series (Prop. 6.1 and 6.2) seem to be new (to be more careful, we did not found them in the literature).

Finally, we would like to point out that the interest for the Klein quartic knew a renewal at the end of the last century, in view of its applications to coding theory and cryptography, starting with the contributions of Hansen [17], Rotillon & Thioung-Ly [32], Duursma [7], and Pellikaan [25]; see also [20].

This article is organized as follows. The definition of Klein’s representation of the group \(G = \text{PSL}(2, \mathbb{F}_7)\) in a three-dimensional space is recalled in section 2. This representation is needed in section 3 in order to state our first main Theorem 3.2, giving the automorphy properties of the set of three Ramanujan’s theta series \(a, b, c\). These series have been rediscovered by Selberg, twenty years later. The transition to three Hecke’s theta series \(x, y, z\) is performed in section 4. Theorem 3.2 implies immediately that \(x, y, z\) give a parametrization of the Klein quartic \(X\), defining an isomorphism from \(X(7)\) onto \(X\); this is done in section 5, followed by a discussion of various other proofs.

The second part of the paper is devoted to subcoverings of the Klein curve, namely, the modular curve \(X_0(7)\), and above all \(X_0(49)\). In any case we need some information on the former, because it is a subcovering of the latter. The curve \(X_0(7)\) is introduced in section 6, and since this curve is rational, any suitably invariant expression of \(x, y, z\) can be expressed in terms of the \textit{hauptmodul} of that curve. This give rise to remarkable identities. Three of the aforementioned formulas of Ramanujan are of this kind: they lead to Klein’s relation giving the explicit formula for the covering \(X(7) \to \mathbb{P}^1\) of degree 168. On the way, we obtain some identities involving Eisenstein series. Then a complete description of the morphisms connecting the three curves is given in section 7.

The third part of the paper is devoted to arithmetical applications. Since \(X_0(49)\) is an elliptic modular curve, there is essentially only one cusp form of weight 2 for \(\Gamma_0(49)\): we show in section 8 that this form can be expressed in terms of \(x, y, z\), and that it is in fact the theta series of a Hecke Grössencharakter. We establish in section
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A quadruple coincidence between four $L$-series, namely, on one side, the $L$-series of that modular cusp form (thus coming from the theory of automorphic forms), and three series on the other side: one associated to a Hecke Grössencharakter (coming from algebraic number theory), one associated to some Jacobi sums (coming from the theory of exponential sums), and eventually the $L$-series of the modular elliptic curve of level 49 (coming from from algebraic geometry), a modest instance of the Langlands program. These identifications give in section 10 some useful informations on the values of the Frobenius endomorphism of the reduction of the Klein quartic over finite fields, and congruence properties of its number of points. We also give in that section the structure (in terms of Hermitian modules) of the Jacobian of the form of the Klein quartic admitting $G$ as a group of automorphisms. This curve has sometimes a maximal or minimal number of points.

2. A Representation of $G$

Felix Klein discovered that the group of automorphisms of $X$ is isomorphic to the simple group $G = PSL(2, \mathbb{F}_7)$ of order 168. One can be more precise. Recall that the principal congruence subgroup of level 7 is the normal subgroup $\Gamma(7)$ of $\Gamma(1) = PSL(2, \mathbb{Z})$ defined by the exact sequence

$$1 \longrightarrow \Gamma(7) \longrightarrow \Gamma(1) \longrightarrow G \longrightarrow 1$$

where $f(\gamma) = \gamma \mod 7$. Then, there is a representation

$$\rho: \Gamma(1) \longrightarrow PGL(3, \mathbb{C})$$

with kernel $\Gamma(7)$ and leaving $\Phi_4$ invariant, defined as follows: if $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and if $\zeta = e^{2i\pi/7}$, then

$$\rho(T) = \begin{bmatrix} \zeta^4 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta \end{bmatrix} \quad \text{and} \quad \rho(S) = \frac{-1}{\sqrt{-7}} \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_2 & \xi_3 & \xi_1 \\ \xi_3 & \xi_1 & \xi_2 \end{bmatrix},$$

with

$$\xi_1 = \zeta - \zeta^6 = 2i \sin \frac{2\pi}{7}, \quad \xi_2 = \zeta^2 - \zeta^5 = 2i \sin \frac{3\pi}{7}, \quad \xi_3 = \zeta^4 - \zeta^3 = -2i \sin \frac{\pi}{7},$$

and $\rho(S)$ is a real symmetric orthogonal matrix.

We should note in passing that $P = (1 : 0 : 0)$ belongs to $X$, as well as

$$\rho(S)P = (\sin \frac{2\pi}{7} : \sin \frac{3\pi}{7} : -\sin \frac{\pi}{7}),$$

and this implies the identity [3, p. 184]:

$$\frac{\sin^2 \frac{\pi}{7}}{\sin \frac{3\pi}{7}} = \frac{\sin^2 \frac{2\pi}{7}}{\sin \frac{\pi}{7}} + \frac{\sin^2 \frac{3\pi}{7}}{\sin \frac{2\pi}{7}} = 0.$$

Recall that $\Gamma(1)$ is defined by the presentation

$$< s, t; s^2 = (st)^3 = 1 >$$

and $S$ and $T$ satisfy these relations. Similarly, $G$ is defined by the presentation

$$< s, t; s^2 = t^7 = (st)^3 = (st^3)^4 = 1 >$$

satisfied by $\rho(S)$ and $\rho(T)$. The group $G$ contains a subgroup of order 3, generated by the image of

$$H = ST^{-3}STST^{-1}ST^3 = \begin{bmatrix} 2 & 7 \\ 7 & 25 \end{bmatrix}.$$
and

\[ \rho(H) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \]

The group \( \Gamma(1) \) operates by homographies on the half-plane \( \mathcal{H} = \{ z \mid \text{Im}(z) > 0 \} \) and its extension \( \mathcal{H}^* \) obtained by including the cusps. The quotient \( Y(7) = \Gamma(7)\backslash \mathcal{H} \) is a Riemann surface since \( \Gamma(7) \) is torsion free, and the modular curve of level 7 is the compact Riemann surface \( X(7) = \Gamma(7)\backslash \mathcal{H}^* \). This Riemann surface has genus 3 and \( G \) acts faithfully on it by construction. From this, one deduces that \( X(7) \) is isomorphic to the Klein quartic \( X \), which is, up to isomorphism, the only Riemann surface of genus 3 on which \( G \) acts faithfully [10, p. 70].

3. Ramanujan’s theta series of order 7

In order to define an explicit isomorphism from \( X(7) \) to \( X \), we start from three theta series: if \( q = \exp(2\pi i z) \) and \( z \in \mathcal{H} \), we define

\[ a(z) = -\sum_{n \in \mathbb{Z}} (-1)^n q^{(14n+5)z^2/14}, \]
\[ b(z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{(14n+3)z^2/14}, \]
\[ c(z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{(14n+1)z^2/14}, \]

and we define an holomorphic map of \( \mathcal{H} \) into \( \mathbb{C}^3 \):

\[ A(z) = \begin{bmatrix} a(z) \\ b(z) \\ c(z) \end{bmatrix}. \]

These series are those of Ramanujan’s definition (D) of the Appendix. We have made a slight change by setting \((a, b, c) = (-w, v, u)\).

3.1. Proposition. If \( z \in \mathcal{H} \), the following relations hold:

\[ A(z + 1) = e^{-\pi/4} \rho(T)A(z), \]
\[ A(-1/z) = e^{3\pi/4} \sqrt{z} \rho(S)A(z), \]

where \( 0 < \arg \sqrt{z} \leq \pi/2 \).

Proof. The theta constants are the functions

\[ \theta \left[ \begin{array}{c} \varepsilon \\ \varepsilon' \end{array} \right] (z) = e^{i\pi \varepsilon \varepsilon'/2} \sum_{n \in \mathbb{Z}} e^{i\pi n \varepsilon' z^2/2}, \]

where \( \varepsilon \) and \( \varepsilon' \) are two real numbers [11, p. 72 sqq]. If \( k \) and \( l \) are integers, then

\[ \theta \left[ \begin{array}{c} l/k \\ 1 \end{array} \right] (kz) = e^{i\pi l/2k} \sum_{n \equiv l \text{ (mod 2k)}} (-1)^n q^{n^2/8k}. \]

If \( k \) is odd, the modified theta constants of order \( k \) are

\[ \varphi_l(z) = \theta \left[ \begin{array}{c} (2l + 1)/k \\ 1 \end{array} \right] (kz), \quad l = 0, \ldots, \frac{k-3}{2}. \]

One checks that the functions \( a, b, c \) are proportional to the three modified theta constants of order 7:

\[ a(z) = -e^{-5\pi/14} \varphi_2(z), \quad b(z) = e^{-3\pi/14} \varphi_1(z), \quad c(z) = e^{-\pi/14} \varphi_0(z). \]

The required relations then follow from the transformation formulas of modified theta constants [11, p. 217]. \( \square \)
We define now
\[ j(\gamma, z) = cz + d \quad \text{if} \quad z \in \mathfrak{H} \text{ and } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1). \]
Recall that if
\[ \eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \]
is the Dedekind eta function, if \( \gamma \in \Gamma(1) \), and if \( z \in \mathfrak{H} \), then [11, p. 339]
\[ \eta(\gamma z) = v_\gamma(\gamma) j(\gamma, z)^{1/2} \eta(z), \]
with a multiplier \( v_\gamma(\gamma) \) independent of \( z \) which is a 24-th root of unity, and where the argument of the square root is in \([0, \pi[)\).

3.2. Theorem. If \((z) \in \mathfrak{H}, \) then
\[ A(\gamma z) = v_\gamma(\gamma)^{-3} j(\gamma, z)^{1/2} \rho(\gamma) A(z) \quad \text{if} \quad \gamma \in \Gamma(1). \]

Proof. Let \( X_1(z) = \eta^3(z) A(z) \). It is sufficient to prove that
\[ X_1(\gamma z) = (cz + d)^2 \rho(\gamma) X_1(z) \quad \text{if} \quad \gamma \in \Gamma(1), \]
and for this it suffices to establish that
\[
\begin{align*}
X_1(z + 1) & = \rho(T) X_1(z), \\
X_1(-1/z) & = z^2 \rho(S) X_1(z).
\end{align*}
\]
But this is a consequence of Proposition 3.1, the formulas
\[ \eta(z + 1)^3 = e^{ix/4} \eta(z)^3, \quad \eta(-1/z)^3 = e^{-3ix/4} z^{3/2} \eta(z)^3, \]
being taken into account. \( \square \)

Since \( \Gamma(7) = \ker \rho \), we deduce from Theorem 3.2 that
\[ A(\gamma z) = v_\gamma(\gamma)^{-3} j(\gamma, z)^{1/2} A(z) \quad \text{if} \quad \gamma \in \Gamma(7), \]
This means that the functions \( a, b, c \) are automorphic forms of weight 1/2 for \( \Gamma(7) \) with the same factor of automorphy. In order to describe the product expansion of \( a, b, c \), we use the standard notation
\[ (a; q) = \prod_{n=0}^{\infty} (1 - aq^n), \]
if \( a \in \mathbb{C}^\times \) and if \( |q| < 1 \), in such a way that \( \eta(z) = q^{1/24}(q; q) \). From the product formula [11, p. 141] we deduce
\[ \theta \left[ \begin{array}{c} 1/k \\ \frac{l}{1/k} \end{array} \right](kz) = \exp(\pi i/2k) q^{l^2/2k} (q^{(k-1)/2}; q^k)(q^{(k+1)/2}; q^k)(q^k; q^k), \]
and the expansions are
\[
\begin{align*}
a(z) & = - q^{25/56} (q; q^7)(q^6; q^7)(q^7; q^7), \\
b(z) & = q^{9/56} (q^2; q^7)(q^7; q^7)(q^7; q^7), \\
c(z) & = q^{1/56} (q^2; q^7)(q^2; q^7)(q^7; q^7).
\end{align*}
\]
This implies
\[ a(z)b(z)c(z) = -\eta(z) \eta(7z)^2, \]
a relation equivalent to Ramanujan's formula (F).
3.3. Remark. If $a \in \mathbb{C}^*$ and if $|q| < 1$, we use the notations $(a;q)_0 = 1$, and

$$(a; q)_n = \prod_{m=0}^{n-1} (1 - a q^m) \quad (n \geq 1).$$

In two articles, [30] and [31], Selberg rediscovered the series $a, b, c$, found the above product expansions, and established the identities

$$a(z) = \sum_{n=0}^{\infty} \frac{(q^2; q^2)_\infty}{(q^2; q^2)_n} \frac{q^{2n(n+1)}}{(-q; -q)_{2n+1}},$$

$$b(z) = \sum_{n=0}^{\infty} \frac{(q^2; q^2)_\infty}{(q^2; q^2)_n} \frac{q^{2n(n+1)}}{(-q; -q)_{2n}},$$

$$c(z) = \sum_{n=0}^{\infty} \frac{(q^2; q^2)_\infty}{(q^2; q^2)_n} \frac{q^{2n^2}}{(-q; -q)_{2n}}.$$

He also set up a link between $a, b, c$ and Ramanujan’s seventh order mock theta functions [29, pp. 127–131].

If $\Gamma$ is a congruence subgroup of $\Gamma(1)$, and if $k$ is a positive integer, we denote by $M_k(\Gamma)$ the space of cusp forms of weight $k$ for $\Gamma$, i.e. satisfying

$$f(\gamma z) = j(\gamma, z)^k f(z) \quad \text{if} \quad \gamma \in \Gamma,$$

and by $S_k(\Gamma)$ the subspace of cusp forms. The above product expansions show that $a, b, c$ vanish at the infinite cusp of $X(7)$, and Theorem 3.2 implies that they vanish at any other cusp, since the operation of $\Gamma(1)$ on the cusps is transitive. Since $q^{1/7}$ is a local parameter for $X(7)$ at the infinite cusp, the eighth powers of $a, b, c$ are cusp forms and belong to $S_4(\Gamma(7))$.

4. Hecke theta series

The ring of integers of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-7})$ is $\mathbb{Z}[\alpha]$, where

$$\alpha = \frac{-1 + \sqrt{7}}{2}.$$

Let $\mathfrak{f} = (\sqrt{-7})$ be the unique prime ideal above 7 in $\mathbb{Z}[\alpha]$. We define three Hecke theta series, where $z \in \mathfrak{f}$ and $\xi \in \mathbb{Z}[\alpha]$:

$$x(z) = - \sum_{\xi \equiv 1(\text{mod } \mathfrak{f})} \xi q^{\xi/7},$$

$$y(z) = \sum_{\xi \equiv 2(\text{mod } \mathfrak{f})} \xi q^{\xi/7},$$

$$z(z) = \sum_{\xi \equiv 4(\text{mod } \mathfrak{f})} \xi q^{\xi/7},$$

and the holomorphic map of $\mathfrak{f}$ into $\mathbb{C}^3$:

$$X(z) = \begin{bmatrix} x(z) \\ y(z) \\ z(z) \end{bmatrix}.$$

4.1. Proposition. The functions $x, y, z$ form a basis of the space $S_2(\Gamma(7))$, and

$$X(z) = \eta^3(z) \mathbb{A}(z).$$
Proof. If \( \xi = x + y\alpha \), then
\[
\xi^\bar{\cdot} = Q(x, y), \quad \text{where} \quad Q(x, y) = x^2 - xy + 2y^2.
\]
Now it follows from Schoeneberg’s theorem [24, p. IV-22] that for any non constant affine function \( L(x, y) \), the series
\[
\sum_{(x, y) \in \mathbb{Z}^2} L(x, y) e^{2\pi i Q(x, y)}
\]
belongs to \( S_2(\Gamma(7)) \). By taking appropriate affine functions, one deduces from this that \( x, y, z \) belongs to \( S_2(\Gamma(7)) \). Hence, the coordinates of \( X - \eta^3 A \) belongs also to \( S_2(\Gamma(7)) \); by checking \( q \)-expansions, one ensures that these cusp forms vanish identically. The \( q \)-expansions show as well the linear independence of these forms, and the dimension of \( S_2(\Gamma(7)) \) is equal to the genus of \( X(7) \), which is 3. \( \square \)

From Theorem 3.2 we deduce:

4.2. Corollary. If \( z \in \mathcal{H} \), then
\[
X(\gamma z) = j(\gamma, z)^2 \rho(\gamma)X(z) \quad \text{if} \quad \gamma \in \Gamma(1). \quad \square
\]

5. Parametrization of the Klein quartic

The first main result is that the cusp forms \( x, y, z \) leads to a parametrization of the Klein quartic:

5.1. Proposition. The cusp forms \( x, y, z \) satisfy
\[
(2) \quad \Phi_4(x, y, z) = 0,
\]
and the map
\[
X(7) \longrightarrow X
\]
deduced from \( X \) by projection is an isomorphism.

Observe that the two holomorphic mappings from \( X(7) \) to \( \mathbb{P}^2 \) deduced from \( X \) and \( A \) by projection are identical, i.e. \( (x : y : z) = (a : b : c) \).

Proof. As in [10]. Corollary 4.2 implies that \( \Phi_4(x, y, z) \) is a cusp form of weight 8 for \( \Gamma(1) \), but the only such form is zero. The linear map \( f(z) \mapsto f(z)dz \) is an isomorphism from \( S_2(\Gamma(7)) \) to the space \( H^0(X(7), \Omega^1) \) of holomorphic differential forms on \( X(7) \), hence, the map \( X \) is the canonical embedding, which is an isomorphism since \( X(7) \) is of genus 3 and not hyperelliptic (recall that we know that \( X(7) \) is isomorphic to the Klein quartic). \( \square \)

The identity (2) in terms of \( (a, b, c) \) is equivalent to Ramanujan’s formula (E). We record here some other proofs.

(i) The classical Jacobi theta series is
\[
\vartheta_1(x, z) = -\theta_1^1(x, z) = -i \sum_{n \in \mathbb{Z}} (-1)^n e^{i\pi(2n+1)x} q^{(2n+1)^2/8}
\]
where we used the notations of [11] in the middle term. The expression of \( A \) in terms of this theta series is
\[
^1A(z) = [e^{i\pi z/7} \vartheta_1(z, 7z), -e^{4i\pi z/7} \vartheta_1(2z, 7z), -e^{9i\pi z/7} \vartheta_1(3z, 7z)].
\]

Proofs of (2) using these notations appears in [3, p. 176] and [6]. We remark in passing that the functions \( x, y, z \) can be fully expressed in terms of theta functions, since [11, p. 289];
\[
\vartheta_1^1(0, z) = 2\pi \eta^3(z).
\]
(ii) If one applies the Frické involution $w_7z = -1/7z$ and makes use of the transformation formula of $\vartheta_1(x, z)$, we obtain:

$$i\Lambda(w_7z) = e^{-\pi i/4} \sqrt{-\vartheta_1(1/7, z), \vartheta_1(2/7, z), \vartheta_1(3/7, z).}$$

Then (2) is equivalent to the same identity applied to the three functions inside the brackets. A proof of this relation, based on elliptic functions, is given in [22].

(iii) Proposition 5.1 is proven in [11, p. 246] by an analysis of the divisors of $a, b, c$ and the identity (2) for these functions is proved again in [11, p. 256] as a particular case of a general three term identity.

(iv) An approach of (2) based on infinite determinants is to be found in [8].

(v) The morphism of Proposition 5.1 is expressed in terms of Klein forms in [16].

6. The rational curve $X_0(7)$

Recall that the Hecke modular curve of level $N$ is the quotient $X_0(N) = \Gamma_0(N) \backslash \mathfrak{H}$ of the extended upper half-space by the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\} \quad (N \geq 1).$$

Then $\Gamma(7)$ is a normal subgroup of $\Gamma_0(7)$, and $\Gamma_0(7)$ is generated by $\Gamma(7)$ and the group generated by $H$ and $T$. The quotient group $\Gamma_0(7)/\Gamma(7)$ is of order 21, isomorphic to the subgroup $g$ of $G$ made of upper triangular matrices. The natural projection defines a Galois covering

$$X(7) \longrightarrow X_0(7)$$

with Galois group $g$. Recall also that

$$j_7(z) = (\eta(z)/\eta(7z))^4$$

is the hauptmodul of $X_0(7)$, that is, the unique univalent modular function for $\Gamma_0(7)$ holomorphic on $\mathfrak{H}$ with a simple pole at $z = i\infty$ and a simple zero at $z = 0$. Let

$$\Phi_3(X, Y, Z) = XYZ, \quad \Phi_6(X, Y, Z) = XY^5 + YZ^5 + ZX^5 - 5X^2Y^2Z^2.$$ 

Then

$$\Phi_3(x, y, z) = -\eta(z)^{10}\eta(7z)^2, \quad \Phi_6(x, y, z) = \Delta(z) = \eta^{24}(z).$$

The first relation comes from (1); for the second, remark that $\Phi_6(x, y, z)$ belongs to the one dimensional space $M_{12}(\Gamma(1))$ since $\Phi_6$ is $G$-invariant [10, p. 57]. We obtain

$$\Phi_6(x, y, z)/\Phi_3(x, y, z)^2 = j_7,$$

and we get a commutative diagram

$$\begin{array}{ccc}
X(7) & \longrightarrow & X_0(7) \\
\sim \downarrow \quad \sim \downarrow j_7 \\
X & \Phi_6/\Phi_3 \quad \rightarrow & \mathbb{P}^1
\end{array}$$

The equality (3) means that

$$\frac{1}{xyz} \left( \frac{x^4}{y} + \frac{y^4}{z} + \frac{z^4}{x} \right) = j_7 + 5,$$

a relation equivalent to Ramanujan’s formula (C). Such an identity is not due to chance: the function field $M_0(\Gamma_0(7))$ is the subfield of $M_0(\Gamma(7))$ left fixed by the Galois group $g$. This means that if $F(X, Y, Z)$ is any rational function, homogeneous of degree zero, invariant under cyclic permutation, and such that

$$F(\zeta^4X, \zeta^2Y, \zeta Z) = F(X, Y, Z),$$


then $F(x, y, z)$ is a rational function of $j_7$, and conversely. More generally, if we only assume that $F$ is degree $k$, then $F(x, y, z)$ is a meromorphic automorphic form of weight $2k$. As a first example, let

$$\Phi_9(X, Y, Z) = X^5Y^4 + Y^5Z^4 + Z^5X^4.$$ 

Since

$$\Phi_3 \Phi_6 + 8 \Phi_3^3 = -\Phi_9 \mod \Phi_4,$$

we obtain

$$-xyz \left( \frac{x}{y^2} \right)^3 = j_7 + 8,$$

a relation equivalent to Ramanujan’s formula (B). A second example is given by Klein’s expression of the absolute invariant $j$ in terms of $j_7$, which can be obtained as follows from Ramanujan’s formula (G). Let

$$\Phi_5(X, Y, Z) = -\left( X^2Y^3 + Y^2Z^3 + Z^2X^3 \right),$$

and

$$P(U, V) = U^2 + 13UV + 49V^2.$$ 

Then the relation

$$\Phi_3^3 = \Phi_3 P(\Phi_6, \Phi_3)^2 \mod \Phi_4$$

implies

$$xyz \left( \frac{x}{y^2} \right)^3 = j_7^2 + 8j_7 + 49,$$

which is equivalent to (G). Let $\Phi_{14}$ be the $G$-invariant form of degree 14 obtained by Klein as a differential determinant [10, p. 58], and

$$Q(U, V) = U^2 + 5.7^2 UV + 7^4 V^2.$$ 

Then

$$\Phi_{14} \Phi_3 \equiv \Phi_5 Q(\Phi_6, \Phi_3^3) \mod \Phi_4.$$ 

From (5) and (6) we deduce

$$\Phi_{14} \Phi_3 \equiv \Phi_5 Q(\Phi_6, \Phi_3) \mod \Phi_4.$$ 

Denote by $E_4$ the classical Eisenstein series of weight 4 for $\Gamma(1)$. Then [10, p. 85]

$$E_4 = \frac{\Phi_{14}(x, y, z)}{\Phi_6(x, y, z)^2}.$$

Since $j = E_4^3/\Delta$, we have

$$j = \frac{\Phi_{14}(x, y, z)^3}{\Phi_6(x, y, z)^7},$$

and dividing both sides of (7) by $\Phi_3^2 \Phi_6^2$, we obtain Klein’s relation

$$j = f_7(j_7), \quad f_7(x) = P(x, 1)Q(x, 1)^3/x^7.$$ 

Let $M_k(\Gamma_0(7), \chi_7)$ be the space of modular forms of type $(k, \chi_7)$, with the character $\chi_7(d) = (-7/d)$: a form in this space satisfies

$$f(\gamma z) = \chi_7(d) (cz + d)^k f(z), \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(7).$$ 

The function

$$k(z) = \frac{\eta(z)^7}{\eta(7z)}$$

belongs to $M_0(\Gamma_0(7), \chi_7)$, and $xyz = -k^2 \mathfrak{t}$, where

$$\mathfrak{t}(z) = (\eta(7z)/\eta(z))^4 = 1/j_7(z).$$
It is an easy exercise to see that Klein’s relation (9) is equivalent to two classical identities in the “Lost Notebook” of Ramanujan [26], which are

\[ E_3^3 = k^3 R(1, t), \quad E_6 = k^2 S(1, t), \]

where \( E_6 \) is the Eisenstein series of weight 6 for \( \Gamma(1) \), and

\[ S(U, V) = U^3 - 2.57^2 U^3 V - 3.7^4 U^2 V^2 - 2.7^6 U V^3 - 7^7 V^4. \]

It is worthwhile to notice that these identities show that \( E_4 \) and \( E_6 \) are given by an algebraic expression involving only \( \eta(z) \) and \( \eta(7z) \). Note also that we shall give in Prop. 6.1 a related identity expressing rationally \( E_4 \) itself in terms of \( x, y, z \) and \( t \).

The series

\[ \Theta_0(z) = \sum_{\xi \in \mathbb{Z}[\alpha]} q^{\xi \xi} \]

belongs to \( M_1(\Gamma_0(7), \chi_7) \) by Schoeneberg’s theorem. Since

\[ R(n) = \{ (x, y) \in \mathbb{Z}^2 \mid x^2 + xy + 2y^2 = n \} = 2 \sum_{d | n} \chi_7(d), \]

we have

\[ \Theta_0(z) = 1 + \sum_{n=1}^{\infty} R(n) q^n = 1 + 2 \sum_{n=1}^{\infty} (\frac{\eta}{\chi_7}) \frac{q^n}{1 - q^n}. \]

6.1. Proposition. We have the double equality

\[ -xyz \left( \frac{x}{y^2} + \frac{y}{z^2} + \frac{z}{x^2} \right) = \Theta_0 k = \frac{j^2}{j^2 + 5.7^2 j^2 + 7^4} E_4. \]

Hence, \( \Theta \) is given by an algebraic expression involving only \( \eta(z) \) and \( \eta(7z) \).

Proof. From (6) we deduce that the first term is equal to the last one. In order to prove the first equality, observe that since \( k \in M_3(\Gamma_0(7), \chi_7) \), both sides are in \( M_4(\Gamma_0(7)) \); check the \( q \)-expansions. \( \square \)

Now \( \Theta_0^2 \in M_2(\Gamma_0(7)) \). Since this space is one dimensional, with basis the Eisenstein series

\[ E_2^{(7)}(z) = \frac{d}{dq} \log t(q) = 1 + 4 \sum_{n=1}^{\infty} \sigma_1(n)(q^n - 7q^{7n}), \]

we have actually \( E_2^{(7)} = \Theta_0^2 \), another identity of Ramanujan [2, Entry 5(i), p. 467]. Moreover

6.2. Proposition. We have

\[ \frac{x^7 + y^7 + z^7}{(xyz)^2} = E_2^{(7)}. \]

Proof. The left member belongs to \( M_2(\Gamma_0(7)) \), with constant term 1. \( \square \)

7. The elliptic curve \( X_0(49) \)

Recall that there is, up to isomorphism, only one elliptic curve \( E \) defined over \( \mathbb{Q} \) with complex multiplication by \( \mathbb{Z}[\alpha] \). The absolute invariant is \( j(E) = j(\alpha) = -15^3 \). A Weierstrass equation of such a curve is

\[ (10) \quad y^2 = x^3 - 35x + 98. \]

The endomorphism of degree 2 providing the multiplication by \( \alpha \) is

\[ (x, y) \mapsto \left( \alpha^{-2}(x - \frac{7(1 - \alpha)^2}{x - \alpha - 4}), \alpha^{-3}y(1 + \frac{7(1 - \alpha)^2}{(x - \alpha - 4)^2}) \right). \]
The curve $E$ has conductor 49, and is listed as the curve $A(7)$ by Gross [14]. Another characterization of this curve is given in Remark 7.3 below.

A morphism from the Klein quartic $X$ to $E$ is defined as follows. Let
\begin{align*}
f &= \frac{x + y}{y} + \frac{z}{x} = \frac{x^2y + y^2z + z^2x}{xyz}, \\
g &= \frac{x + y}{z} + \frac{z}{x} = \frac{xy^2 + yz^2 + zx^2}{xyz}.
\end{align*}
(11)

These expressions are invariant under cyclic permutations of the letters and define two functions on $X$, with poles located at the points $(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)$, of respective orders 2 for $f$ and 3 for $g$. By looking at the local expansions of these functions at these poles, one finds for these functions the relation
\begin{equation}
g^2 - 3fg + g + f^3 - 2f + 3 = 0,
\end{equation}
which defines a curve isomorphic to $E$ (the absolute invariant is the same), with discriminant $\Delta = -7^3$. Namely, if
\begin{align*}
f &= -1 - \frac{1}{u}, \\
g &= -\frac{3}{2} \left( \frac{v}{u} + \frac{u}{v} \right) - 2,
\end{align*}
(13)
then $(u, v)$ belongs to the projective affine curve $E$ with equation
\begin{equation}
v^2 = 28u^3 + 21u^2 + 4u,
\end{equation}
which is a third model of $E$, since we recover (10) from (14) by the change of variables
\begin{align*}
u &= \frac{x - 7}{28}, \\
v &= \frac{y}{28}.
\end{align*}
Hence, the couple $(u, v)$ defines a dominant morphism
\begin{equation}
\varphi : X \longrightarrow \mathcal{E}
\end{equation}
which identifies $\mathcal{E}$ with the quotient of $X$ by the group of order 3 of cyclic permutations on three letters.

This morphism has a modular counterpart. If $f(z)$ is a modular form of weight $k$ for $\Gamma_0(N^2)$, then $\alpha^* f(z) = f(z/N)$ is a modular form of the same weight for the subgroup
\begin{equation}
\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}) \mid b \equiv c \equiv 0 \pmod{N} \right\},
\end{equation}
and conversely. Since $\Gamma(N) \subset \Gamma_0(N)$, we get a dominant morphism
\begin{equation}
X(N) \longrightarrow X_0(N^2).
\end{equation}
If $N = 7$, then the curve $X_0(49)$ is of genus one, and is isomorphic to $E$. The construction of this isomorphism is obtained as follows. Firstly, we note that $\Gamma_0(7)$ is generated by $\Gamma(7)$ and the group generated by $H$, whose elements performs cyclic permutations of $x, y, z$. Then, substituting the forms $x, y, z$ to the variables $x, y, z$ in the relations (11) yields three forms which are modular for $\Gamma_0(7)$, because they are modular for $\Gamma(7)$ and for the group generated by $H$. Now, If $f(z)$ is a modular form for $\Gamma_0(7)$, then $f_7(z) = f(7z)$ is modular for $\Gamma_0(49)$. Hence, $(x^2y + y^2z + z^2x)_7$, $(xy^2 + yz^2 + zx^2)_7$, $(xyz)_7$ are three cusp forms of weight 6 for $\Gamma_0(49)$, and the two expressions
\begin{align*}
f &= (\frac{x}{z} + \frac{y}{x} + \frac{z}{y})_7, \\
g &= (\frac{x}{y} + \frac{y}{z} + \frac{z}{x})_7,
\end{align*}
define modular functions on $X_0(49)$. Then the functions $u$ and $v$ defined by
\begin{align*}
f &= -1 - \frac{1}{u}, \\
g &= -\frac{3}{2} \left( \frac{3}{u} + \frac{v}{u^2} \right) - 2,
\end{align*}
(15)
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satisfy (14), from which we deduce:

7.1. Proposition. The map

\[ G : X_0(49) \longrightarrow \mathcal{E} \]

defined by \( G(z) = (u(z), v(z)) \) is an isomorphism. \( \square \)

This is the simplest case of modular theory, which ensures that if \( E \) is an elliptic curve defined over \( \mathbb{Q} \) of conductor \( N \), there exists a surjective map \( X_0(N) \longrightarrow E \). There is a unique such map of minimal degree (up to composing with automorphisms of \( E \)), and its degree is the modular degree of \( E \). Here Prop. 7.1 shows that \( E \) is of modular degree one. The following diagram commutes:

\[
\begin{array}{ccc}
X(7) & \xrightarrow{\alpha} & X_0(49) \\
\downarrow & & \downarrow G \\
\mathfrak{X} & \xrightarrow{\varphi} & \mathcal{E}
\end{array}
\]

We now express \( u \) and \( v \) in terms of \( \eta(z) \), and for this purpose we use two functions defined by Fricke [12, p. 400-403]. The first one is \( t(z) = (\eta(7z)/\eta(z))^4 \), and the second one is \( \eta(49z)/\eta(z) \). By comparing \( q \)-expansions, one proves that

\[ u(z) = \eta(49z)/\eta(z). \]

Then \( t \) and \( u \) generates the function field of \( X_0(49) \): the comparison of \( q \)-expansions leads to

\[ v = \frac{2t - 7u(1 + 5u + 7u^2)}{1 + 7u + 7u^2}. \]  

The two preceding identities, together with (15), give an expression of \( f \) and \( g \) in terms of the eta function; considering the relation

\[ \frac{v}{u^2} = \sqrt{\frac{4}{u^3} + \frac{21}{u^3} + \frac{28}{u}}, \]

we see that the second formula of (15) is equivalent to Ramanujan’s formula (A), and that the first one is equivalent to a formula of the Notebooks [2, Entry 17(v), p. 303].

7.2. Remark. The Fricke involution \( w_{49}z = -1/(49z) \) is an automorphism of \( X_0(49) \) permuting the two cusps, and

\[ t(w_{49}z) = \frac{t(z)}{49u(z)}, \quad u(w_{49}z) = \frac{1}{7u(z)}, \quad v(w_{49}z) = \frac{v(z)}{7u(z)^2}, \]

hence, \( w_{49} \) corresponds to the involutive automorphism \( P \mapsto P_0 - P \) of \( \mathcal{E} \), where \( P_0 = (0, 0) \), given by

\[ (u, v) \mapsto \left( \frac{1}{7u}, \frac{v}{7u^2} \right). \]

The expression (16) of \( v \) leads to

\[ 2t = 7u(1 + 5u + 7u^2) + (1 + 7u + 7u^2)\sqrt{28u^3 + 21u^2 + 4u}, \]

which is also equivalent to a formula of the Notebooks [2, Entry 18(vi), p. 306]. This equation was first proved by Watson in [39], and is called the modular relation for the prime 7 [11, p. 392 and 399]. If we write (17) as \( t = \psi(u,v) \), we get
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\[ j(z) = 1/\psi \circ G(z), \]
and the various diagrams combine themselves in order to give the full picture

\[
\begin{array}{cccc}
X(7) & \xrightarrow{3} & X_0(49) & \xrightarrow{7} \\
\sim & X & \sim & G \\
& \sim & j & \sim \ j
\end{array}
\]

where the degrees of the coverings are indicated on the top horizontal arrows. This gives an explicit factorization of the “Equation of degree 168” of Klein, that is, the covering of degree 168 of the projective plane by the Klein quartic given by (8).

7.3. Remark. Let \( E_0 \) be an ordinary elliptic curve over \( \mathbb{F}_p \), together with its \( p \)-th power Frobenius endomorphism \( F_0 \in \text{End}_{\mathbb{F}_p}(E_0) \), defined over \( \mathbb{F}_p \). Up to isomorphism there is a unique elliptic curve \( E \) over \( \mathbb{Z} \) such that \( E_0 \) is the reduction of \( E \) modulo \( p \) and such that the map

\[ \text{End}_{\mathbb{F}_p}(E_0) \rightarrow \text{End}_{\mathbb{Z}}(E) \]

is an isomorphism [33]. The curve \( E \) is the canonical lift of \( E_0 \); in particular, there is a unique endomorphism \( F \in \text{End}_{\mathbb{Z}}(E) \) mapping to \( F_0 \). Assume now that \( E \) is our elliptic curve with CM by \( \mathbb{Z}[\alpha] \). In the equation of \( E \) given by (12), let \( f = 1-x \) and \( g = 1-2x-y \); we get

\[ y^2 + xy = x^3 - x^2 - 2x - 1, \]

which reduces modulo 2 as the equation of an ordinary elliptic curve \( E_0 \) over \( \mathbb{F}_2 \):

\[ y^2 + xy = x^3 + x^2 + 1. \]

Since the roots of the Frobenius automorphism \( F \) of \( E_0 \) are \( -\alpha \) and \( -\overline{\alpha} \), we have

\[ F^2 - F + 2 = 0, \]

hence, the ring \( \text{End}_{\mathbb{F}_p}(E_0) \) is isomorphic to \( \mathbb{Z}[\alpha] \) and \( E \) is the canonical lift of \( E_0 \).

8. Hecke series and cusp forms

The inclusion \( \mathbb{Z} \hookrightarrow \mathbb{Z}[\alpha] \) induces a ring isomorphism

\[ \mathbb{Z}/7\mathbb{Z} \cong \mathbb{Z}[[\alpha]]/(f), \]

for instance \( \alpha \equiv 3 (\text{mod} \ f) \). Composing the inverse isomorphism with the quadratic residue symbol gives a character on \( \mathbb{Z}[\alpha] \), given, for \( \xi \in \mathbb{Z}[\alpha] \), by

\[ (\xi \ f)_2 \begin{cases} 
1 & \text{if } \xi \text{ is a non-zero square } (\text{mod } f), \\
-1 & \text{if } \xi \text{ is a non-square } (\text{mod } f), \\
0 & \text{if } \xi \equiv 0 \text{ (mod } f). 
\end{cases} \]

We denote by \( I(f) \) the group of the fractional non-zero ideals in \( K \) whose factorization in prime ideals does not involve \( f \). If \( \xi \in K^\times \), define

\[ \lambda(\xi) = (\xi \ f)_2 \xi. \]

This is an even function which depends only of the ideal generated by \( \xi \). Thus, if \( a \in I(f) \) is generated by \( \xi \), we set \( \lambda(a) = \lambda(\xi) \). The map

\[ \lambda : I(f) \rightarrow K^\times \]

is a Hecke Grössencharakter of \( K \) with conductor \( f \), according to the usual criterion [41]. We associate to this Grössencharakter its Theta series

\[ \Theta(z, \lambda) = \sum_{a \in I(f)} \lambda(a) q^{N(a)} \]
where \( N(a) \) is the norm of the ideal \( a \). We frequently write \( \Theta(z, \lambda) = \Theta(z) \) for simplicity. This Theta series has a \( q \)-expansion

\[
\Theta(z) = \sum_{n=1}^{\infty} c_n q^n
\]

whose first terms are

\[
\Theta(z) = q + q^2 - q^4 - 3q^8 - 3q^9 + 4q^{11} - q^{16} + 4q^{22} + 8q^{23} - 5q^{25} + 5q^{29} + 3q^{36} - 6q^{37} + O(q^{43}).
\]

8.1. **Proposition.** The cusp forms \( x, y, z \) satisfy

\[
(x + y + z)^7 = \Theta,
\]

and this function is the unique form of the one dimensional space \( S_2(\Gamma_0(49)) \) such that \( \Theta(q) \sim q \) if \( q \to 0 \). Moreover

\[
\Theta(w_{49} z) = -49 z^2 \Theta(z).
\]

**Proof.** The Grössencharakter \( \lambda \) is an isomorphism of \( I(f) \) to the subgroup of \( K^\times \) made of squares modulo \( f \), hence

\[
\Theta(z) = \frac{1}{2} \sum_{\xi \in \mathbb{Z}[a]} \xi q^{\xi^2} = \sum_{\xi \equiv 1, 2, 4(\text{mod } f)} \xi q^{\xi^2},
\]

and this implies the first formula. Now we know that \((x + y + z)_7 \) belongs to the space \( S_2(\Gamma_0(49)) \), which is one-dimensional, since \( X_0(49) \) is an elliptic curve. Since the sum of the columns of \( \rho(S) \) is equal to \(-1\), we deduce from Cor. 4.2 that the modular form \( s_1(z) = x(z) + y(z) + z(z) \) satisfies

\[
s_1(-1/z) = -z^2 s_1(z),
\]

from which the second formula follows. \( \square \)

One can express \( \Theta \) in terms of classical modular forms on \( \Gamma_0(49) \). A standard modular form of weight 2 for \( \Gamma_0(49) \) is the Eisenstein series

\[
E_2^{(49)}(q) = q \frac{d}{dq} \log \frac{\eta(q^{49})}{\eta(q)} = 2 + \sum_{n=1}^{\infty} \sigma_1(n)(q^n - 49q^{49n}),
\]

where \( \sigma_1(n) \) is the sum of divisors of \( n \). Now \( \Theta_0^{(7)}(7z) \) is a modular form of weight 2 for \( \Gamma_0(49) \), and it is easy to see that \( 4E_2^{(49)}(z) = E_2^{(7)}(z) + 7E_2^{(7)}(7z) \), hence

\[
4E_2^{(49)}(z) = \Theta_0(z)^2 + 7\Theta_0(7z)^2.
\]

8.2. **Proposition.** We have

\[
\Theta = \frac{u}{v} E_2^{(49)}.
\]

Hence, \( \Theta \) is given by an algebraic expression involving only \( \eta(z), \eta(7z), \) and \( \eta(49z) \).

**Proof.** The normalized invariant differential form on the elliptic curve \( E \) is

\[
\omega = \frac{du}{v},
\]

from which one deduces that

\[
\Theta = v^{-1} q \frac{du}{dq},
\]

and since

\[
E_2^{(49)}(q) = q \frac{d}{dq} \log u(q) = u^{-1} q \frac{du}{dq},
\]

one gets the required formula. \( \square \)
9. L-series

The L-series of \( \lambda \) is defined if \( \text{Re}(s) > 1 \) by
\[
L(\lambda, s) = \sum_{a \in I(f)} \lambda(a) N(a)^{-s} = \prod_{p \neq \mathfrak{f}} (1 - \lambda(p) N(p)^{-s})^{-1}.
\]

This is as well the L-series of the modular form \( \Theta \):
\[
(19) \quad L(\lambda, s) = L(\Theta, s) = (2\pi)^s \Gamma(s)^{-1} \int_{0}^{\infty} \Theta(iy, \lambda) y^{s-1} dy,
\]
where \( \Gamma(s) \) is Euler's Gamma function. From the relation \( \Theta(w_{49} z) = -49 z^2 \Theta(z) \), one deduces the functional equation
\[
7^{2-s} \hat{L}(\lambda, 2-s) = 7^s \hat{L}(\lambda, s), \quad \text{where} \quad \hat{L}(\lambda, s) = (2\pi)^s \Gamma(s)^{-1} \int_{0}^{\infty} \Theta(iy, \lambda) y^{s-1} dy.
\]

Now let \( p \) be an odd prime number \( \neq 7 \). If \( (p/7) = +1 \) then \( (p) = \mathfrak{p} \mathfrak{p}^2 \) with a prime \( p \) of \( K \), thus \( N(p) = p \), and there is a unique integer \( \pi_p \) generating \( p \) such that
\[
\pi_p = a_p + b_p \sqrt{-7}, \quad a_p^2 + 7b_p^2 = p, \quad \left( \frac{a_p}{7} \right) = 1, \quad b_p \neq 0,
\]
with \( a_p \) and \( b_p \) in \( \mathbb{Z} \), because \( p \) is odd; note that \( a_p = \text{Re} \pi_p = \text{Re} \pi_p^\alpha \) depends only on \( p \). In that case
\[
\lambda(p) = \pi_p.
\]
If \( p = 2 \), we take \( \pi = (\alpha) \) since \( \alpha \bar{\alpha} = 2 \), and we define \( \pi_p = -\bar{\alpha} \). If \( (p/7) = -1 \) then \( p = \mathfrak{p} \) is inert in \( K \), thus \( N(p) = p^2 \), and
\[
\lambda(p) = -p.
\]

9.1. Remark. Let \( p \) be a prime number with \( p \equiv 1 \pmod{7} \). Then \( (p/7) = +1 \) and \( (p) = \mathfrak{p} \mathfrak{p}^2 \). Let \( q \) be a prime ideal of \( L = \mathbb{Q}(\zeta_7) \) above \( p \). For every \( x \in \mathbb{Z}[\zeta_7] \), prime to \( q \), the 7-th power residue symbol \( (x/q)_7 \) is the unique 7-th root of unity satisfying the condition
\[
x^{(p-1)/7} \equiv \left( \frac{x}{q} \right)_7 \pmod{q}.
\]

The 7-th power residue symbol induces a multiplicative character \( \chi_q \) of order 7 of the field \( \kappa = \mathbb{Z}[\zeta]/q \) with 7 elements. We are now able to define the Jacobi sum
\[
J(q) = -\sum_{x \in \kappa} \chi_q(x(1-x)^2).
\]

Recall the following results [20]: the Stickelberger congruence Relation implies that \( (J_q) = \mathfrak{p} \), hence, the value of these sums depends only on \( \mathfrak{p} \), and we denote their common value by \( J_p \). Then the Ihara congruence Relation implies
\[
J(p) = \lambda(p),
\]
in other words, the Grössencharakter \( \lambda \) is defined by the Jacobi sum \( J_p \), a particular case of Weil’s Theorem [40]. Thusage the L-series of this family of exponential sums, namely
\[
L(J, s) = \prod_{p \neq \mathfrak{f}} (1 - J(p) N(p)^{-s})^{-1},
\]
and the L-series of the Grössencharakter are identical:
\[
L(J, s) = L(\lambda, s).
\]

Since
\[
L(\lambda, s) = \sum_{a \in I(f)} \lambda(a) N(a)^{-s} = \sum_{n=1}^{\infty} c_n n^{-s},
\]

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one has
\[ c_p = \text{Tr} \lambda(p) = \text{Tr} \pi_p = 2a_p \quad \text{if } (p/7) = +1, \]
\[ = 0 \quad \text{if } (p/7) = -1. \]

From the product expansion we deduce that
\[ L(\lambda, s) = \prod_{p \neq 7} (1 - c_p p^{-s} + p^{1-2s})^{-1}. \]

9.2. **Remark.** The Chebotarev density theorem [21, Ch. VIII] tells us that the set \( S \) of primes \( p \) such that \( (p/7) = -1 \) has density \( 1/2 \). Since \( p \in S \) if and only if \( c_p = 0 \), the set of integers \( n \in \mathbb{N} \) such that \( c_n \neq 0 \) has density 0 [34, Th. 13].

The elliptic curve \( E \) has good reduction for any prime \( p \neq 7 \). If \( p \) is such a prime, we denote by \( E/\mathbb{F}_p \) its reduction modulo \( p \), and by \( P(E/\mathbb{F}_p, T) \) the numerator of the zeta function of \( E/\mathbb{F}_p \). The global \( L \)-series of \( E \) is then
\[ L(E/\mathbb{Q}, s) = \prod_{p \neq 7} P(E/\mathbb{F}_p, p^{-s})^{-1}. \]

9.3. **Proposition.** If \( p \neq 7 \), then
\[ P(E/\mathbb{F}_p, T) = 1 - c_p T + pT^2, \quad \text{and} \quad |E(\mathbb{F}_p)| = p + 1 - c_p. \]

Moreover
\[ L(E/\mathbb{Q}, s) = L(\lambda, s). \]

**Proof.** The two assertions are equivalent. Since \( E = X_0(49) \), Shimura’s theorem [38, p.182] implies that \( L(E, s) \) is obtained from the Mellin transform of the unique normalized form in \( S_2(\Gamma_0(49)) \):
\[ (2\pi)^{-s} \Gamma(s)L(E, s) = \int_0^\infty \Theta(iy) y^{s-1} dy; \]

now compare with (19). \( \square \)

9.4. **Remarks.** (i) A direct proof of Proposition 9.3 can be obtained following the method for CM elliptic curves explained in [37, Ch. II, § 10]. This method is applied to the particular elliptic curve \( E \) in [14] and [27].

(ii) Since \( c_p = 0 \) if and only if \( (p/7) = -1 \), the set \( S \) is the set of primes for which \( E/\mathbb{F}_p \) is supersingular by Proposition 9.3.

(iii) As observed by Shimura [38, p. 221], Formula (20) implies that \( L(E, 1) \neq 0 \). In fact, according to Gross and Zagier [15], one has
\[ L(1, \chi) = \frac{1}{4\pi \sqrt{7}} \Gamma(\frac{1}{7})^2 \Gamma(\frac{2}{7}) \Gamma(\frac{4}{7}). \]

This fact is in agreement with the conjecture of Birch and Swinnerton-Dyer, since \( E(\mathbb{Q}) \) has only two points: \((0,0)\) and the point at infinity [10, p. 73].

The following result provides recurring relations for \( c_{p^n} \) and \( |E(\mathbb{F}_{p^n})| \).

9.5. **Corollary.** let \( T_n \) be the \( n \)-th Hecke operator. If \( (n, 7) = 1 \), then \( T_n \Theta = c_n \Theta \).

Moreover, if \( n \geq 2 \), and if \( p \neq 7 \), then
\[ c_{p^n} = c_{p^{n-1}} c_p - pc_{p^{n-2}}, \]
and
\[ |E(\mathbb{F}_{p^n})| = p^n + 1 - c_{p^n} + pc_{p^{n-2}}. \]
Proof. Since

\[ L(\Theta, s) = \sum_{n=1}^{\infty} c_n n^{-s} = \prod_{p \neq 7} (1 - c_p p^{-s} + p^{1-2s})^{-1}, \]

the series \( \Theta \) is a common eigenvalue of the Hecke operators [38, Th. 3.43]. The recurrence relation for \( c_p \) is a consequence of the basic properties of Hecke operators, and this relation implies

\[ \frac{1}{P(E/\mathbb{F}_p, T)} = \frac{1}{1 - c_p T + pT^2} = \sum_{n=0}^{\infty} c_{p^n} T^n, \]

from which one deduces

\[ -\frac{P'}{P}(E/\mathbb{F}_p, T) = c(p) + \sum_{n=2}^{\infty} (c_{p^n} - pc_{p^n-1})T^{n-1}. \]

On the other hand, by definition of the zeta function of \( E/\mathbb{F}_p \), one has

\[ -\frac{P'}{P}(E/\mathbb{F}_p, T) = \sum_{n=1}^{\infty} (p^n - 1 - |E(\mathbb{F}_p^n)|)T^{n-1}, \]

and the last assertion follows. \( \square \)

9.6. Corollary. One has \(|E(\mathbb{F}_p)| = p + 1\) if \( p \equiv 3, 5, 6(\text{mod } 7) \), and

\(|E(\mathbb{F}_p)| = 0, 16, 8(\text{mod } 28) \) if \( p \equiv 1, 2, 4(\text{mod } 7) \) respectively \((p \neq 2)\).

Proof. We already saw the first property. Regarding the second, it follows from the definition of \( a_p = \text{Re} \pi_p \) that

\[ c_p \equiv 2, 1, 4(\text{mod } 7) \quad \text{if } p \equiv 1, 2, 4(\text{mod } 7). \]

From this and Proposition 9.3 we deduce

\[ |E(\mathbb{F}_p)| \equiv 0, 2, 1(\text{mod } 7) \quad \text{if } p \equiv 1, 2, 4(\text{mod } 7), \]

Now observe that a model of \( E \) is given by the equation

\[ y^2 = 4x^3 + 21x^2 + 28x. \]

Here, the group \( E[2] \) of points of order 2 belongs to \( E(K) \), since

\[ 4x^3 + 21x^2 + 28x = 4x(x - \sqrt{-7} \frac{\alpha^4}{4})(x + \sqrt{-7} \frac{\alpha^4}{4}), \]

and \( E[2] \) reduces to a subgroup of order 4 of \( E(\mathbb{F}_p) \). \( \square \)

10. Jacobians and reduction over finite fields

In this section, if \( X \) is a curve defined over \( k \) and if \( k' \) is an extension of \( k \), we denote by \( X/k' \) the curve over \( k' \) obtained from \( X \) by extension of scalars.

The quartic \( \Phi_4 \) has discriminant \( 314.77 \), hence, the Klein quartic has good reduction for any prime different from 3 and 7. We denote by \( X/k \) the curve defined over a field \( k \) with equation \( \Phi_4 = 0 \) (the coefficients being assumed in \( k! \)). First, we borrow some information on the number of points of \( X/k \), when \( k \) is a finite field. We denote by \( P(X/k, T) \) the numerator of the zeta function of a curve \( X/k \).
10.1. Proposition. If \( q \not\equiv 1(\text{mod } 7) \), then 
\[
|\mathcal{X}(\mathbb{F}_q)| = q + 1.
\]
If \( p \equiv 3 \text{ or } 5 \)(mod 7), then \( P(\mathcal{X}/\mathbb{F}_p, T) = 1 + p^2 T^6 \), and 
\[
|\mathcal{X}(\mathbb{F}_p)| = p^m + 1 + 6p^{m/2} \text{ if } m \equiv 0 \text{ (mod } 6).\]
If \( p \equiv 6 \)(mod 7), then \( P(\mathcal{X}/\mathbb{F}_p, T) = (1 + pT^2)^3 \) and 
\[
|\mathcal{X}(\mathbb{F}_p)| = p^m + 1 + 6p^{m/2} \text{ if } m \text{ is even.}
\]

**Proof.** Denote by \( \mathcal{F}_7 \) the Fermat curve of degree 7. The map 
\[
f(x : y : z) = (x^3 z : y^3 x : z^3 y)
\]
is a dominant morphism from \( \mathcal{F}_7 \) to \( \mathcal{X} \), and the map 
\[
g(x : y : z) = (x^3 y : y^3 z : z^3 x)
\]
is a dominant morphism from \( \mathcal{X} \) to the line \( D \) with equation \( x + y + z = 0 \). Then 
\[
g \circ f(x : y : z) = (x^7 : y^7 : z^7).
\]
If \( q \not\equiv 1(\text{mod } 7) \), then \( \lambda \mapsto \lambda^7 \) is an permutation of \( \mathbb{F}_q \), hence, \( g \circ f \) induces a permutation of \( \mathbb{P}^2(\mathbb{F}_q) \), and this proves the first equality. The map 
\[
h(x : y : z) = (x^3 y : xyz^2 : -z^3 x)
\]
induces an isomorphism defined over \( k \) from \( \mathcal{X} \) to the curve 
\[
y^7 = x(z - x)^2 z^4.
\]
Now the two other assertions are consequences of the Stickelberger Relation, as it is proved in [13, Lem. 1.1] or [18, Ex. 12, p. 226]. \( \square \)

If \( k \) is a finite field, the various \( k \)-forms of \( \mathcal{X}/k \) are the curves \( \mathcal{Y}/k \) defined over \( k \) which are isomorphic to \( \mathcal{X}/k \) over the algebraic closure of \( k \). The \( k \)-forms \( \mathcal{Y}/k \) reflecting at the best the properties of the Klein quartic will be those such that \( \text{Aut}_k(\mathcal{Y}/k) = G \). Since \( \text{Aut}_{\mathbb{C}}(\mathcal{X}/\mathbb{C}) \) is defined over the cyclotomic field \( L = \mathbb{Q}(\zeta) \) of level 7, it is worthwhile to start from another curve isomorphic over \( L \) to the Klein quartic, namely the quartic \( \mathcal{C} \) defined over \( K \) with equation found by Ciani [5, p. 369]
\[
\Psi_4(x, y, z) = 0,
\]
where 
\[
\Psi_4(x, y, z) = x^4 + y^4 + z^4 + 3\alpha(x^2 y^2 + y^2 z^2 + z^2 x^2).
\]
This choice is more convenient, since Ciani proved that the group \( \text{Aut}_{\mathbb{C}}(\mathcal{C}/\mathbb{C}) \) is defined over \( K \). This equation is the one coming from Elkies ‘‘\( S_4 \) model’’, and the generators of \( \text{Aut}_{\mathbb{C}}(\mathcal{C}/\mathbb{C}) \) are [10, p. 55]
\[
\rho(S) = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \rho(T) = \frac{1}{2} \begin{bmatrix} -1 & 1 & \bar{\alpha} \\ \alpha & \alpha & 0 \\ -1 & 1 & -\bar{\alpha} \end{bmatrix},
\]
from which one verifies at once Ciani’s assertion. Recall that an isomorphism 
\( \mathcal{X}/L \to \mathcal{C}/L \) is provided by 
\[
T = \begin{bmatrix} \zeta^6 + \zeta^2 & \alpha \zeta + 1 & 1 \\ 1 & \zeta^6 + \zeta^2 & \alpha \zeta + 1 \\ \alpha \zeta + 1 & 1 & \zeta^6 + \zeta^2 \end{bmatrix},
\]

since \( \Psi_4(T(x, y, z)) = c \Phi_4(x, y, z) \) with a suitable \( c \in L \).
Now let \( k = \mathbb{F}_q \), where \( q \) is a prime power. Remark that there is a couple of distinct solutions \( \alpha, \bar{\alpha} \) of the equation \( x^2 + x + 2 = 0 \) in \( k \) if and only if \((q/7) = 1\), that is, if \( q \) is a square modulo 7. If that is so, one defines successively an isomorphism \( G \overset{\sim}{\rightarrow} \text{Aut}_k(\mathcal{E}/k) \subset \text{PGL}(3, k) \) by reduction of the isomorphism \( G \overset{\sim}{\rightarrow} \text{Aut}_c(\mathcal{E}/\mathbb{C}) \) in characteristic 0, and a plane quartic \( \mathcal{C}/k \) by reduction of the Ciani quartic \( \mathcal{C} \), which has good reduction for any prime different from 7.

We shall now show that if \( q \) is a square modulo 7, then \( \text{Jac} \mathcal{C}/k \) is isogenous to \((E/k)^3\) (for any curve \( X \), we denote by \( \text{Jac} X \) the Jacobian of \( X \), endowed with its natural polarization). One can even be more precise if one uses Hermitian modules in the style of Serre [36]. Let \( \text{Mod}(\mathbb{Z}[\alpha]) \) be the category of couples \( (M, H) \), where \( M \) is an indecomposable \( \mathbb{Z}[\alpha] \)-module of finite type without torsion, and where \( H \) is a positive-definite Hermitian form, such that \( M \) is of discriminant 1. We write \( M \) in place of \((M, H)\).

Serre proved [23, pp. 235–236] that there is an isomorphism

\[
\text{Aut}_k(\mathcal{C}/k) = \{ \pm 1 \} \times \text{Aut}_k(X/k).
\]

Then \( \text{Aut}(P) = G \), and moreover \( P \) is the unique module of rank 3 in \( \text{Mod}(\mathbb{Z}[\alpha]) \), see [10, p. 61]. Turning back to finite fields, one knows [36] that for any curve \( X/k \) such that \( \text{Jac} X/k \) is isogenous to \((E/k)^3\), there is a unique \( M \in \text{Mod}(\mathbb{Z}[\alpha]) \) such that

\[
\text{Jac} X/k \sim (E/k) \otimes_{\mathbb{Z}[\alpha]} M.
\]

In order to find a curve fulfilling these properties, the curve \( X/k \) is not convenient. In fact, according to Torelli’s theorem, see [35] or [23, pp. 236], for any curve \( X \) defined over \( k \), one has

\[
\text{Aut}_k(\text{Jac} X/k) = \{ \pm 1 \} \times \text{Aut}_k(X/k).
\]

Hence, \( \text{Aut}_k(X/k) \) must be equal to \( G \) and the related representation of \( G \) must be defined over \( k \).

10.2. Theorem. Let \( k = \mathbb{F}_q \), where \( q \) is a square modulo 7, and \( \mathcal{C}/k \) the Ciani quartic with equation \( \Psi_4 = 0 \). Then \( \text{Aut}_k(\mathcal{C}/k) \) is isomorphic to \( G \), and

\[
\text{Jac} \mathcal{C}/k \sim (E/k) \otimes_{\mathbb{Z}[\alpha]} P,
\]

In particular \( \text{Jac} \mathcal{C}/k \) is isogenous to \((E/k)^3\). Moreover, if \( q \equiv 1(\text{mod } 7) \), then \( \mathcal{C}/k \) is isomorphic to the reduction \( X/k \) of the Klein quartic.

Proof. First of all, \( \text{Jac} \mathcal{C}/k \) is isogenous to \((E/k)^3\) : an isogeny is explicitly constructed in [1], or one can use alternately [9, Prop. 2]. This implies, as explained above, that

\[
\text{Jac} \mathcal{C}/k \sim (E/k) \otimes_{\mathbb{Z}[\alpha]} M',
\]

where \( M' \in \text{Mod}(\mathbb{Z}[\alpha]) \). Now \( M \) is of rank 3 and \( G \subset \text{Aut} M' \), but the only such module is \( P \), as we know. Finally, if \( q \equiv 1(\text{mod } 7) \), then \( \mu_7 \subset k \) and the reduction of the matrix \( T \) above is in \( k \).

Theorem 10.2 has the following consequences. Let \( k = \mathbb{F}_p \) where \( p \) is prime and a square modulo 7. Since there are fast algorithms for the computation of \( \pi_k \), we have thus obtained a way to calculate fastly the number of points of the Ciani quartic over a finite field:
10.3. Corollary. Let \( k = \mathbb{F}_q \), where \( q \) is a square modulo 7. Then
\[
P(\mathbb{C}/k, T) = P(E/k, T)^3 = |1 - \pi_p T|^6.
\]
Hence,
\[
|\mathcal{C}(\mathbb{F}_q)| = p + 1 - 3 \text{Tr}(\pi_p) = p + 1 - 3c_p.
\]
and if \( n \geq 2 \), then
\[
|\mathcal{C}(\mathbb{F}_q^n)| = p^n + 1 - 3 \text{Tr}(\pi_p^n) = p^n + 1 - 3(c_p^n - p\alpha_{p^n-2}).
\]
Moreover if \( q \equiv 1(\text{mod } 7) \), then \(|\mathcal{C}(\mathbb{F}_q)| = |\mathcal{C}(\mathbb{F}_q)|. \)

From Corollary 9.6 we deduce some congruences for that number of points:

10.4. Corollary. One has
\[
|\mathcal{C}(\mathbb{F}_p)| \equiv 24, 0, 0 \pmod{28} \quad \text{if } p \equiv 1, 2, 4 \pmod{7}. \]

Considering the prime powers \( \leq 1000 \), with 7 excluded, the maximal number of points is attained in the following cases:

\[
\begin{array}{c|cccccccc}
q & 4 & 8 & 43 & 107 & 151 & 169 & 683 & \cdots & 2^{13} \\
|\mathcal{X}(\mathbb{F}_q)| & 4 & 24 & 80 & 168 & 224 & 248 & 840 & \cdots & 8736 \\
\end{array}
\]

and the minimal number of points in the following cases:

\[
\begin{array}{c|cccccccc}
q & 2 & 11 & 23 & 32 & 71 & 263 & 331 & 491 & 907 \\
|\mathcal{X}(\mathbb{F}_q)| & 0 & 0 & 0 & 0 & 24 & 168 & 224 & 360 & 728 \\
\end{array}
\]

One can prove, by a method of Serre [10, p. 77], that \( \mathcal{C}/\mathbb{F}_q \) is, for every \( q \) in the above displays, the unique curve of genus 3 over \( \mathbb{F}_q \) with that number of points. The proof runs as follows. Let \( \mathcal{X}'/\mathbb{F}_q \) be such a curve; since it is maximal (say), it has the same Frobenius eigenvalues than \( \mathcal{C}/\mathbb{F}_q \), hence,
\[
\text{Jac } \mathcal{X}'/\mathbb{F}_q \sim (E/k) \otimes \mathbb{Z}[\alpha] M'
\]
with \( M' \in \text{Mod}(\mathbb{Z}[\pi_p]) \). But one checks that \( \mathbb{Z}[\pi_p] = \mathbb{Z}[\alpha] \) in these cases, hence, \( M' = P \) since \( P \) is the only member of \( \text{Mod}(\mathbb{Z}[\alpha]) \). Then apply Torelli’s Theorem.

**Appendix A. Ramanujan’s Formulas Verbatim**

For the convenience of the reader, we reproduce below the last eight lines of [28], p. 300, as edited by Berndt [3]. Recall Ramanujan’s notation:

\[
f(a, b) = \sum_{n=-\infty}^{\infty} a^n b^{n-1/2}, \quad |ab| < 1,
\]
and
\[
f(-x) = f(-x, -x^2) = \sum_{n=-\infty}^{\infty} (-1)^n x^{3n-1/2}, \quad |x| < 1.
\]

The passage begins here (the labels and the equalities inside the brackets have been added):

\[
(A) \quad \frac{1}{x^{3/4}} \frac{f(-x^3, -x^4)}{f(-x, -x^6)} - 2 - x^{1/4} \frac{f(-x^3, -x^4)}{f(-x^3, -x^4)} + x^{3/4} \frac{f(-x, -x^6)}{f(-x^3, -x^4)} = \frac{1}{2} \left\{ \frac{3f(-x^{1/7})}{x^{2/7} f(-x^2)} + \frac{4f^3(-x^{1/7})}{x^{2/7} f(-x^2)} + \frac{21f^2(-x^{1/7})}{x^{2/7} f(-x^2)} + \frac{28f(-x^{1/7})}{x^{2/7} f(-x^2)} \right\}
\]

\[
(B) \quad \frac{\alpha^2}{\gamma^2} + \frac{\beta^2}{\alpha} - \frac{\gamma^2}{\beta} = 8 + \frac{f^4(-x)}{x f^4(-x)}
\]

\[
(C) \quad \frac{\alpha^2}{\gamma^2} - \frac{\beta^2}{\alpha} - \frac{\gamma^2}{\alpha^2} = 5 + \frac{f^4(-x)}{x f^4(-x)}
\]
(D) \[ u = x^{1/56} f(-x^3, -x^4), \quad v = x^{9/56} f(-x^2, -x^5), \quad w = x^{25/56} f(-x, -x^6) \]

then

(E) \[ \frac{u^2}{v} - \frac{v^2}{w} + \frac{w^2}{u} = 0 \]

(F) \[ uvw = x^{5/8} f(-x) f^2(-x) \]

(G) \[ \frac{v}{u^2} + \frac{w}{u^2} + \frac{u}{w^2} = \frac{f(-x)}{x^{2/5} f^2(-x^7)} \sqrt{\frac{f^4(-x)}{f^4(-x^7)}} + 13x + 49x^2 \frac{f^4(-x^7)}{f^4(-x)} \]

References

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Institut de Mathématiques de Luminy
Luminy Case 907
13288 Marseille Cedex 9
FRANCE