LEFSCHETZ THEOREMS AND DEPENDENT RATIONAL POINTS
ON CURVES OVER FINITE FIELDS.

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Abstract. For a smooth curve $C$ over a finite field $\mathbb{F}_q$, we prove that the probability that a randomly chosen set of $\tau$ rational points impose dependent conditions on a given linear system of dimension $\tau$ is asymptotically equal to $\frac{1}{q^\tau}$.

The proof involves a geometric construction and a Lefschetz theorem for quasi-projective varieties.

The result has applications in the assessment of the performance of decoding algorithms for algebraic geometry codes.

Let $C$ be a smooth and absolutely irreducible curve of genus $g$ defined over the finite field $\mathbb{F}_q$ and let $D$ be a $\mathbb{F}_q$-rational divisor on $C$ with $l(D) = \tau$.

Let $X$ be $\tau$-tuples of pairwise different points on $C$, i.e.,

$$X = \{(P_1, \ldots, P_\tau) \mid P_i \neq P_j \text{ for } i \neq j\}$$

and let $\Gamma \subset X$ be $\tau$-tuples of pairwise different points on $C$ failing to impose independent conditions on the linear system of divisors equivalent to $D$. Specifically, if $\mathbb{F}_q(C)$ denotes the field of rational functions on $C$, then

$$\Gamma = \{(P_1, \ldots, P_\tau) \in X \mid \exists f \in \mathbb{F}_q(C) : \text{div}(f) + D - (P_1 + \ldots + P_\tau) \geq 0\}.$$ 

Let $|X(\mathbb{F}_q)|$ and $|\Gamma(\mathbb{F}_q)|$ denote the number of $\mathbb{F}_q$-rational points on $X$ and $\Gamma$. Then we prove that

Theorem 1. In the notation above assume that $\text{deg}(D) \geq 2g + 1$ and let $\tau = \text{deg}(D) + 1 - g$. Assume $\Gamma \neq \emptyset$. There is a constant $c$ (independent of $j$), such that

$$|X(\mathbb{F}_q)| - q^j |\Gamma(\mathbb{F}_q)| \leq c (q^j)^{\frac{\tau + 1}{2}} .$$

The bounding term $c (q^j)^{\frac{\tau + 1}{2}}$ cannot in general be replaced by a smaller power of $q^j$, as the following example show.

Example 2. Let $C$ be an elliptic curve with $|C(\mathbb{F}_q)| = 1 + q$ and let $D = 3P_0$. Then $\tau = 3$ and $\Gamma$ is triples of collinear points on $C$. In this case we have

$$|X(\mathbb{F}_q)| = |C(\mathbb{F}_q)| (|C(\mathbb{F}_q)| - 1) (|C(\mathbb{F}_q)| - 2) = q^3 - q$$

$$|\Gamma(\mathbb{F}_q)| = (|C(\mathbb{F}_q)| - 9) (|C(\mathbb{F}_q)| - 1 - 4) = (q - 8) (q - 4) = q^2 - 12q + 32$$


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assuming that the 2-torsion and 3-torsion points are $\mathbb{F}_q$-rational. This follows from the fact that 3 points on $C$ are collinear if and only if they have sum 0 in the group structure on the elliptic curve. We now have for all uneven $j$, that

$$|X(\mathbb{F}_q)| - q |\Gamma(\mathbb{F}_q)| = -12(q^j)^2 - 36q^j.$$ 

A result of the above type has applications in the assessment of the performance of decoding algorithms for algebraic geometry codes according to [JNH].

Central to the proof of the theorem is the following lemma, which is obtained through a geometric construction.

**Lemma 3.** In the notation above

i) $X \setminus \Gamma$ is affine.
ii) $\Gamma$ is smooth if $\deg(D) \geq 2g + 1$

**Proof.** Let $(a_{i,1}: \ldots: a_{i,\tau})$ be homogeneous coordinates on the $i$’th copy of $\mathbb{P}^{\tau-1}$ in $\mathbb{P}^{\tau-1} \times \ldots \times \mathbb{P}^{\tau-1}$ and let $V \subset \mathbb{P}^{\tau-1} \times \ldots \times \mathbb{P}^{\tau-1}$ be the closed subscheme defined by the vanishing of the determinant

$$\begin{vmatrix}
  a_{1,1} & \cdots & a_{\tau,1} \\
  a_{1,2} & \cdots & a_{\tau,2} \\
  \vdots & \ddots & \vdots \\
  a_{1,\tau} & \cdots & a_{\tau,\tau}
\end{vmatrix}$$

Consider for a moment the Segre embedding

$$\mathbb{P}^{\tau-1} \times \ldots \times \mathbb{P}^{\tau-1} \xrightarrow{\text{Segre}} \mathbb{P}^N, \quad N = \tau! - 1$$

the morphism defined by

$$(a_{1,1}: \ldots: a_{1,\tau}) \times \ldots \times (a_{\tau,1}: \ldots: a_{\tau,\tau}) \mapsto (\ldots: a_{1,i_1} \cdot a_{2,i_2} \cdots \cdot a_{\tau,i_\tau}: \ldots).$$

Then we see, that $V \subset \mathbb{P}^{\tau-1} \times \ldots \times \mathbb{P}^{\tau-1}$ is the inverse image of a hyperplane $H \in \mathbb{P}^N$.

By assumption $\deg(D) \geq 2g+1$, therefore $\tau = l(D) = \deg(D) + 1 - g$ by Riemann-Roch, and the divisor $D$ defines an embedding of the curve $C$ as a smooth curve in $\mathbb{P}^{\tau-1}$:

$$\phi : C \to \mathbb{P}^{\tau-1}.$$

By the definition of $X$ and $\Gamma$, we have that $(P_1, \ldots, P_\tau)$ is in $\Gamma$ if and only if $\phi(P_1), \ldots, \phi(P_\tau)$ are linear dependent in $\mathbb{P}^{\tau}$, equivalently lie in a hyperplane $L \subset \mathbb{P}^{\tau}$, therefore we have the cartesian diagrams of intersections:

$$X \xrightarrow{\tau\text{-fold}} C \times \ldots \times C \xrightarrow{\phi \times \ldots \times \phi} \mathbb{P}^{\tau-1} \times \ldots \times \mathbb{P}^{\tau-1} \xrightarrow{\text{Segre}} \mathbb{P}^N$$

$$\Gamma \xrightarrow{\tau\text{-fold}} (\phi \times \ldots \times \phi)^{-1}(V) \xrightarrow{\tau\text{-fold}} V \xrightarrow{\tau\text{-fold}} H$$

and we note the important fact that

$$X \setminus \Gamma = C \times \ldots \times C \setminus (\phi \times \ldots \times \phi)^{-1}(V).$$

It follows that $X \setminus \Gamma$ is isomorphic to the complement of a hyperplane section in a projective variety and therefore affine, which was the first assertion.
As for assertion on smoothness, assume to the contrary that $(p_1, \ldots, p_τ) ∈ Γ$ is a singular point on $Γ$, this implies that $H$ (and thereby $V$) do not intersect $X$ transversally at $(p_1, \ldots, p_τ)$.

Let $L$ be a hyperplane in $\mathbb{P}^{τ-1}$ through $p_1, \ldots, p_τ$, which exist as $(p_1, \ldots, p_τ) ∈ Γ$. All $τ$-tuples of points in $L$ are linear dependent, i.e. for all $j$, therefore we have

\[
L_j := p_1 × \ldots × p_{j-1} × L × p_{j+1} × \ldots × p_τ ⊂ V ⊂ \mathbb{P}^{τ-1} × \ldots × \mathbb{P}^{τ-1}.
\]

Consider the Cartesian diagrams of intersections in $\mathbb{P}^{τ-1} × \ldots × \mathbb{P}^{τ-1}$:

\[
\begin{array}{ccc}
X & \longrightarrow & \mathbb{P}^{τ-1} × \ldots × \mathbb{P}^{τ-1} \\
\uparrow & & \uparrow \\
Γ & \longrightarrow & V \\
\uparrow & & \uparrow \\
p_1 × \ldots × p_{j-1} × L ∩ C × p_{j+1} × \ldots × p_τ & \longrightarrow & L_j
\end{array}
\]

As the intersection between $X$ and $V$ isn’t transversal at $(p_1, \ldots, p_τ)$, the intersection between $X$ and $p_1 × \ldots × p_{j-1} × L × p_{j+1} × \ldots × p_τ$ can’t be either, consequently $L$ is a tangent hyperplane to the curve $C$ at $p_j$. This is true for all $p_1, \ldots, p_τ$, i.e., there exists a rational functions in $L(D)$ vanishing to at least second order at $p_1, \ldots, p_τ$, therefore $l(D - (2p_1 + \ldots + 2p_τ)) > 0$, however this contradicts the assumption as

\[
\deg(D - (2p_1 + \ldots + 2p_τ)) = \deg(D) - 2l(D)
\]

\[
= \deg(D) - 2(\deg(D) + 1 - g)
\]

\[
= 2g - 2 - \deg(D) < 0.
\]

Assume that the prime $l$ is different from the characteristic of the ground field. Let $\mathbb{Q}_l$ denote the $l$-adic numbers. For a constructible sheaf $\mathcal{F}$ of $\mathbb{Q}_l$-vector spaces $H^i(X, \mathcal{F})$ (resp. $H^i_\text{ét}(X, \mathcal{F})$) denote the étale $l$-adic cohomology groups (resp. the étale $l$-adic cohomology groups with compact support), see [M].

Finally for an integer $c$ we denote by $\mathcal{F}(c)$ the Tate twist of $\mathcal{F}$ and

\[
H^i(X, \mathcal{O}_l(c)) = H^i(X, \mathcal{O}_l(c)) ⊗ \mathbb{Q}_l(c)
\]

The second main ingredient in the proof is a Lefschetz Theorem for quasi-projective varieties. We have not been able to find a reference for it and gives a proof along the lines of [J, Corollaire 7.2], see also [G-L] for related results.

**Lemma 4. A Lefschetz Theorem for quasi-projective varieties.** Let $X ⊂ \mathbb{P}^N$ be a quasi-projective, smooth scheme of dimension $n$ and let $Y = X \cap H$ be a smooth hyperplane section, such that $X \setminus Y$ is affine. Then there are isomorphisms:

\[
H^{i-2}_c(Y, \mathbb{Q}_l(-1)) \rightarrow H^i_c(X, \mathbb{Q}_l)
\]

for $i ≥ n + 2$.

**Proof.** For any locally constant sheaf $\mathcal{F}$ of $\mathbb{Z}/(l)$-modules, the inverse image morphisms:

\[
H^i(X, \mathcal{F}) \rightarrow H^i(Y, \mathcal{F})
\]
are isomorphisms for \( i \leq n - 2 \) as follows from the long exact cohomology sequence using the assumption that \( X \setminus Y \) is affine. As both \( X \) and \( Y \) are assumed to be smooth, Poincaré duality applied to (2) gives the result.

We are ready to prove Theorem 1.

Proof. The ground field is the finite field \( \mathbb{F}_q \) and \( H^i_c(X, \mathbb{Q}_l) \) is endowed with an action of the Frobenius morphism \( \text{Frob} \). The Lefschetz trace formula [M, p.292] by A. Grothendieck determines the number of \( \mathbb{F}_q \)-rational points in terms of the traces of \( \text{Frob} \) on the étale cohomology spaces.

We have accordingly

\[
|X(\mathbb{F}_q)| = \sum_{i=0}^{2\tau} (-1)^i \text{Tr}(\text{Frob} | H^i_c(X, \mathbb{Q}_l)) \tag{3}
\]

\[
q |\Gamma(\mathbb{F}_q)| = q \sum_{i=0}^{2\tau-2} (-1)^i \text{Tr}(\text{Frob} | H^i_c(\Gamma, \mathbb{Q}_l)) \tag{4}
\]

As for the high dimensions, we obtain from Lemma 4 applied to \( X \) and \( \Gamma \), that

\[
q \sum_{i=\tau}^{2\tau-2} (-1)^i \text{Tr}(\text{Frob} | H^i_c(\Gamma, \mathbb{Q}_l)) = \sum_{i=\tau}^{2\tau-2} (-1)^i \text{Tr}(\text{Frob} | H^i_c(\Gamma, \mathbb{Q}_l(-1))) =
\]

\[
\sum_{i=\tau+2}^{2\tau} (-1)^i \text{Tr}(\text{Frob} | H^i_c(X, \mathbb{Q}_l))
\]

Combining this with (3) and (4) gives:

\[
|X(\mathbb{F}_q)| - q |\Gamma(\mathbb{F}_q)| =
\]

\[
\sum_{i=0}^{\tau+1} (-1)^i \text{Tr}(\text{Frob} | H^i_c(X, \mathbb{Q}_l)) - q \sum_{i=0}^{\tau-1} (-1)^i \text{Tr}(\text{Frob} | H^i_c(\Gamma, \mathbb{Q}_l))
\]

Deligne’s main theorem [D] gives that the eigenvalues of \( \text{Frob} \)’s action on the \( i \)’th cohomology group have absolute values \( \leq q^{\frac{i}{2}} \). This immediately implies (1) of Theorem 1 as the dimensions on the cohomology groups do not depend on the power \( j \) of \( q \) and the highest power of \( q \) being \( q^{\frac{\tau+1}{2}} \).

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References


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