Linear codes with exponentially many light vectors

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Let C be a code over $_q$ of length n and distance $d = d(C)$. The (Hamming) distance distribution of the code is an $(n+1)$-vector $(A_0 = 1, A_1, \ldots, A_n)$, where $A_w = A_w(C) := (\sharp C)^{-1}|\{(x, x') \in C^2 : \mathrm{d}(x, x') = w\}|$. Of course $A_w = 0$ if $1 \le w \le d-1$.

Let $\{C_n\}$ be a family of binary linear codes of growing length n and let $d_n = d(C_n)$. G. Kalai and N. Linial [1] conjectured that for any such family the number A_{d_n} is subexponential in n, i.e., that for any $\alpha > 0$ there is a number $N(\alpha)$ such that for all $n > N(\alpha)$ we have $\log A_{d_n} \le \alpha n$ (if the base of logarithms is missing, it is 2 throughout). They also made a similar conjecture about unrestricted (i.e., not necessarily linear) codes and wrote "The [asymptotic] distance distribution near the minimum distance remains a great mystery."

While we now know a little more about the distance distribution of codes in general, this claim is still very much true. The above conjectures, however, are not. Let

$$E_q(\delta) := H(\delta) - \frac{\log q}{\sqrt{q}-1} - \log\frac{q}{q-1},$$

where $H(y) = -y\log y - (1-y)\log(1-y)$. The function $E_q(\delta)$ has two zeros $0 < \delta_1 < \delta_2 < (q-1)/q$ and is positive for $\delta_1 < \delta < \delta_2$.

Theorem 1 *For any $q = 2^{2s}, s = 3, 4, \ldots,$ there exists a sequence of binary linear codes $\{C_n\}$ of length $n = qN, N \to \infty$ and distance $d_n = n\delta/2, \delta_1 < \delta < \delta_2$ such that*

$$\log A_{d_n} \ge N E_q(\delta) - o(N). \tag{1}$$

References

[1] G. Kalai and N. Linial, *On the distance distribution of codes*, IEEE Trans. Inform. Theory (1995), no. 5, 1467–1472.

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