Isogeny classes and Frobenius roots statistics for abelian varieties over finite fields

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Let \( I(g,q,N) \) be the number of isogeny classes of \( g \)-dimensional abelian varieties over a finite field \( \mathbb{F}_q \) having a fixed number \( N \) of \( \mathbb{F}_q \)-rational points. We describe the asymptotic (for \( q \to \infty \)) distribution of \( I(g,q,N) \) over possible values of \( N \). We also prove an analogue of the Sato-Tate conjecture for isogeny classes of \( g \)-dimensional abelian varieties.

Let \( g \) be a fixed positive integer, \( A \) be an abelian variety over a finite field \( \mathbb{F}_q \) of dimension \( g \), and let

\[
X_{A,g} = \{ e^{2\pi i \theta_1}, e^{2\pi i \theta_2}, \ldots, e^{2\pi i \theta_g} \}
\]

be the corresponding set of Frobenius eigenvalues,

\[
\Theta_{A,g} = (\theta_1, \theta_2, \ldots, \theta_g) \in \Sigma_g := \{ \theta \in \mathbb{R}^g : \theta_1 \leq \theta_2 \leq \ldots \leq \theta_g \}.
\]

One defines the (countable) family \( \Xi_g \subseteq \Sigma_g \) by

\[
\Xi_g := \bigcup_{A,g} \Theta_{A,g},
\]

where \( A \) runs over isogeny classes of \( g \)-dimensional abelian varieties over \( \mathbb{F}_q \) such that the multiplicity of any element in \( \Xi_g \) equals one (i.e., it is just a set, which follows from the Honda-Tate theorem). Our main result on the distribution of the Frobenius roots is

**Theorem A.** The set \( \Xi_g \subseteq \Sigma_g \) is uniformly distributed on \( \Sigma_g \) with respect to the probabilistic measure

\[
\nu_{g,n} := \frac{1}{g^n} \left( \prod_{j<k} (\cos \pi \theta_j - \cos \pi \theta_k) \right) \cdot \prod_i (\sin \pi \theta_i \cdot d\theta_i),
\]

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the real positive constant \( v_g \) is given by

\[
v_g = \frac{2^g}{g!} \prod_{j=1}^{g} \left( \frac{2^j}{2^j - 1} \right)^{2^{j-1} - j}.
\]

Other question which we consider is: how many isogeny classes of \( g \)-dimensional abelian varieties over a finite field correspond to the given number of rational points?

The question makes sense due to well-known result by A. W. H. Weil:

**Theorem.** If \( A \) and \( B \) are two isogenous abelian varieties over a finite field \( \mathbb{F}_q \) then

\[
\mathcal{A}(\mathbb{F}_q) = \mathcal{B}(\mathbb{F}_q).
\]

We shall investigate this problem asymptotically, for growing \( q \). More precisely, we fix the dimension \( g \geq 1 \) and for a given prime power \( q \) construct a discrete measure \( \mu_{g,\infty} \) on the interval \([-1, 1]\) characterizing the distribution of the number of isogeny classes of \( g \)-dimensional abelian varieties over \( N \), and then tend \( q \) to infinity. It turns out that there exists the limit measure \( \mu_{g,\infty} \) it has a continuous density, which can be often calculated explicitly.

Our principal result is

**Theorem B.** Let \( g \geq 2 \). The limit measure \( \mu_{g,\infty} \) on \([-1, 1]\) is given by

\[
\mu_{g,\infty} = \mathcal{F}_g(t) dt,
\]

where the function \( \mathcal{F}_g(t) \) satisfies the following conditions:

1. \( \mathcal{F}_g(t) \) is continuous, even, \( \mathcal{F}_g(-1) = \mathcal{F}(1) = 0 \), and \( \mathcal{F}_g(t) > 0 \) for \( t \in (-1, 1) \);

2. on each segment \([-1 + \frac{1}{g^{i-1}}, -1 + \frac{1}{g^i}], i = 1, 2, \ldots, g \) the function \( \mathcal{F}_g(t) \) is given by a polynomial \( P_{g,i}(t) = P_{g,i}(-t) \in \mathbb{Q}[t] \) of degree \( d_{g,i} \leq \frac{2^{2^{i-1}+1}}{g^i} \);

3. \( P_{g,i}(t) = P_{g,i}(-t) = a_g (-1 - t)^{g-1} / (2^{i-1} / g^i) \), with \( a_g \in \mathbb{Q} \);

4. \[
\int_{-1}^{1} \mathcal{F}_g(t) dt = \mathcal{F}_g \cdot \prod_{j=1}^{g} \left( \frac{2^j}{2^j - 1} \right)^{2^{j-1} - j}.
\]