A heuristic explanation of the limiting distribution of Frobenius eigenvalues for principally-polarized abelian varieties over finite fields

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We provide an exact formula for the number of principally-polarized abelian varieties that lie in certain isogeny classes over finite fields. We combine this formula with some heuristic class-number estimates and an argument of Vladut to provide a heuristic explanation of the recent results of Katz and Sarnak on the distribution of Frobenius eigenvalues of principally-polarized abelian varieties.

Let us state our result more precisely. Let $q$ be a power of a prime. Suppose that $f$ is an irreducible polynomial of degree $2n$ whose middle coefficient is coprime to $q$ and whose complex roots all have magnitude $\sqrt{q}$. Then $f$ corresponds (via the Honda-Tate theorem) to an isogeny class $C$ of simple $n$-dimensional ordinary abelian varieties over $\mathbb{F}_q$. Let $\pi$ be a root of $f$ in $\mathbb{Q}$, let $K$ be the CM-field $\mathbb{Q}(\pi)$, let $K^+$ be the maximal real subfield of $K$, and let $R$ be the order $\mathbb{Z}[\pi, \bar{\pi}]$ of $K$.

**Theorem.** Suppose that $K/K^+$ is ramified at a finite prime, that the unit group of $K$ is equal to the unit group of $K^+$, and that the ring $R$ is equal to the maximal order of $K$. Then the number of isomorphism classes of pairs $(A, \lambda)$, where $A$ is an abelian variety in $C$ and $\lambda$ is a principal polarization of $A$, is equal to the quotient $h(K)/h(K^+)$ of the class number of $K$ by the class number of $K^+$.

Let the complex roots of $f$ be $\sqrt{q^{2n+1}}$ for $j = 1, \ldots, n$. If we make the heuristic approximation

$$(\text{class number}) \cdot \text{(regulator)} \approx \sqrt{\text{discriminant}}$$

and calculate the discriminants of $K$ and of $K^+$ in terms of the $\theta_j$, we find that the quotient $h(K)/h(K^+)$ should (heuristically) be on the order of a constant times a certain power of $q$ times

$$\prod_j \sin \theta_j \prod_{j < k} (\cos \theta_j - \cos \theta_k).$$

Combining this heuristic estimate with a result of Vladut that gives the limiting distribution of Frobenius eigenvalues of isogeny classes of abelian varieties over finite fields, we recover the Katz-Sarnak result that the limiting distribution of Frobenius eigenvalues of principally-polarized abelian varieties over a finite field is equal to a constant times

$$\prod_j \sin^2 \theta_j \prod_{j < k} (\cos \theta_j - \cos \theta_k)^2 d\theta_1 \cdots d\theta_n.$$