A heuristic explanation of the limiting distribution of Frobenius eigenvalues for principally-polarized abelian varieties over finite fields

Everett W. Howe

Center for Communications Research, 4320 Westerra Court, San Diego, CA 92121

17 May 2001

We provide an exact formula for the number of principally-polarized abelian varieties that lie in certain isogeny classes over finite fields. We combine this formula with some heuristic class-number estimates and an argument of Vladut to provide a heuristic explanation of the recent results of Katz and Sarnak on the distribution of Frobenius eigenvalues of principally-polarized abelian varieties.

Let us state our result more precisely. Let q be a power of a prime. Suppose that f is an irreducible polynomial of degree $2n$ whose middle coefficient is coprime to q and whose complex roots all have magnitude $\sqrt{q}$. Then f corresponds (via the Honda-Tate theorem) to an isogeny class $\mathcal{C}$ of simple n-dimensional ordinary abelian varieties over $\mathbf{F}_q$. Let π be a root of f in $\overline{\mathbf{Q}}$, let K be the CM-field $\mathbf{Q}(\pi)$, let K^+ be the maximal real subfield of K, and let R be the order $\mathbf{Z}[\pi, \overline{\pi}]$ of K.

Theorem. *Suppose that K/K^+ is ramified at a finite prime, that the unit group of K is equal to the unit group of K^+, and that the ring R is equal to the maximal order of K. Then the number of isomorphism classes of pairs (A, λ), where A is an abelian variety in $\mathcal{C}$ and λ is a principal polarization of A, is equal to the quotient $h(K)/h(K^+)$ of the class number of K by the class number of K^+.*

Let the complex roots of f be $\sqrt{q}e^{\pm i\theta_j}$ for $j = 1, \ldots, n$. If we make the heuristic approximation

$$(\text{class number}) \cdot (\text{regulator}) \approx \sqrt{\text{discriminant}}$$

and calculate the discriminants of K and of K^+ in terms of the θ_j, we find that the quotient $h(K)/h(K^+)$ should (heuristically) be on the order of a constant times a certain power of q times

$$\prod_j \sin\theta_j \prod_{j<k} (\cos\theta_j - \cos\theta_k).$$

Combining this heuristic estimate with a result of Vladut that gives the limiting distribution of Frobenius eigenvalues of isogeny classes of abelian varieties over finite fields, we recover the Katz-Sarnak result that the limiting distribution of Frobenius eigenvalues of principally-polarized abelian varieties over a finite field is equal to a constant times

$$\prod_j \sin^2\theta_j \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_1 \cdots d\theta_n.$$