ABOUT THE DISTRIBUTION OF FROBENIUS ANGLES OF CURVE OF GIVEN GENUS

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$[0, \frac{\pi}{4}]$ $[\frac{\pi}{4}, \frac{\pi}{2}]$ $[\frac{\pi}{2}, \frac{3\pi}{4}]$ $[\frac{3\pi}{4}, \pi]$

1 Introduction

Let X be a curve of genus g, $\mathcal{A} = \{\Theta_i\}_{1 \le i \le g}$ the Frobenius's angles of X. Let q be a prime, $r = \sqrt{q}$ and for $n \in \mathbf{N}^*$, N_n the number of points over the field $\mathbf{F}_q$. So :

$$(1) \quad \forall n \in \mathbf{N}^*, \ N_n r^{-n} = r^n + r^{-n} - 2 \sum_{\Theta \in \mathcal{A}} \cos(n\Theta)$$

Let $(u_n)_{n \in \mathbf{N}}$ be a sequence. We define

$$\psi : \mathbf{R} \longrightarrow \mathbf{R} \quad \text{and} \quad f : \mathbf{R} \longrightarrow \mathbf{R}$$
$$x \longmapsto \sum_{n \ge 1} u_n r^n \qquad \Theta \longmapsto u_0 + 2 \sum_{n \ge 1} u_n \cos(n\Theta)$$

So the sum of $u_n * (1)$ give, with $N = N_1$:

$$(2) \quad N\psi(r^{-1}) + \sum_{n \ge 1} (N_n - N) u_n r^{-n} = u_0 g + \psi(r) + \psi(r^{-1}) - \sum_{\Theta \in \mathcal{A}} f(\Theta)$$

2 Finite case

We can use (2) with hypothesis

Classic case (Serre, 1983)

$$u_0 = 1, \qquad \forall n \ge 1, \ u_n \ge 0, \qquad \forall \Theta \in \mathcal{A}, \ f(\Theta) \ge 0.$$

So we deduce :$N < \frac{g}{\psi(r^{-1})} + 1 + \frac{\psi(r)}{\psi(r^{-1})}$

Case (I)

$$u_0 = 1, \qquad \forall n \ge 1, \ u_n \ge 0, \qquad \exists \Gamma \subset [0, \pi] / \ \forall \Theta \in \Gamma, \ f(\Theta) \ge 0.$$

Every curve with (I) $N > \frac{g}{\psi(r^{-1})} + 1 + \frac{\psi(r)}{\psi(r^{-1})}$ must have the sum $\sum_{\Theta \in \mathcal{A}} f(\Theta)$ negative, and so there is a Frobenius's angle in $[0, \pi] - \Gamma$.

The function $f(x) = \frac{2}{2-\sqrt{2}} \cos(2x) + \frac{2\sqrt{2}-2}{2-\sqrt{2}} \cos(x) + 1$ is not negative over $[0, \frac{\pi}{3}] \bigcup [\frac{3\pi}{4}, \pi]$. For $q = 2$, we find that a curve with $N > \frac{8-2\sqrt{2}}{7} g + \frac{27+2\sqrt{2}}{7}$ has a Frobenius's angle in $]\frac{\pi}{3}, \frac{3\pi}{4}[$. (This inequality applies to a genus between 2 and 28.)

Case (II)

$$u_0 = 0, \qquad \forall n \ge 1, \ u_n \ge 0, \qquad \exists \Gamma \subset [0, \pi] / \ \forall \Theta \in \Gamma, \ f(\Theta) \ge 0.$$

Every curve with (II) $N > 1 + \frac{\psi(r)}{\psi(r^{-1})}$ must have the sum $\sum_{\Theta \in \mathcal{A}} f(\Theta)$ negative, and so there is a Frobenius's angle in $[0, \pi] - \Gamma$.

The function $f(x) = \cos(2x)$ is not negative over $[0, \frac{\pi}{4}] \bigcup [\frac{3\pi}{4}, \pi]$. We find that a curve with $N > 1 + n^4$ has a Frobenius's angle in $]\frac{\pi}{4}, \frac{3\pi}{4}[$. (There are curves with $N > 1 + n^4$.)

Case (III)

$$u_0 < 0, \qquad \forall n \geq 1,\ u_n \geq 0, \qquad \exists \Gamma \subset [0, \pi] /\ \forall \Theta \in \Gamma,\ f(\Theta) \geq 0.$$

If $\psi(r^{-1}) > 0$, then every curve with $(III)\ \ N > \frac{u_0}{\psi(r^{-1})} g + 1 + \frac{\psi(r)}{\psi(r^{-1})}$ must have the sum $\sum_{\Theta \in \mathcal{A}} f(\Theta)$ negative, and so there is a Frobenius's angle in $[0, \pi] - \Gamma$.

A general example

Look at :

$f : \mathbf{R} \longrightarrow \mathbf{R}$

$\quad x \longmapsto \prod_{k=0}^{n-3} \big(\cos(x) - \cos((2k+1)\pi) \big)$.

f is not negative in $[0, \frac{\pi}{n}] \bigcup [\frac{3\pi}{n}, \pi]$. We can write $f(x) = \sum_{k=0}^{n} c_k \cos(kx)$, we have $c_0 = c_1 = 0,\ c_n = c_2,\ c_{n-1} = c_3, \cdots$ and $\forall k \leq n,\ c_k \geq 0$.So :

If $N > 1 + r^n$, there is a Frobenius's angle in $]\frac{\pi}{n}, \frac{3\pi}{n}[$.

3 Infinite case

Let B_n be the number of points of degree n on X.

We suppose the following conditions :

$$\exists (a, A, b, B, C) \in \mathbf{R} \times \mathbf{R_+}^4 /\ \forall n \in \mathbf{N}^*,\ u_n \geq -AC^{-n} r^{-an} \ and \ \frac{B_n}{g} \leq Br^{bn}$$

with $\frac{r^{-1}}{C} < 1$. So we have :

$$\sum_{n \geq 1} (N_n - N) u_n r^{-n} \geq ABgCr^{a+1} \frac{r^b - 2Cr^{a+1}}{Cr^{a+1} - r^b} \frac{r^b}{C^2 r 2(a+1) - 1} = bi(g)$$

And if $N > \frac{bi(g)}{\psi(r^{-1})} + 1 + \frac{\psi(r)}{\psi(r^{-1})}$, there is an angle in $[0, \pi] - \Gamma$

4 Open problems

1) Find other examples for the case (II) then for (III), and compare.

2) Find examples for the infinite case (good choice of a,A,b,B).

3) Find an upper bound of the length of an interval without Frobenius's angle.

4) Find a lower bound of the maximum dimension of a component.

References

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