ABOUT THE DISTRIBUTION OF FROBENIUS ANGLES OF CURVE OF GIVEN GENUS

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\[ [0, \frac{\pi}{4}] \quad [\frac{\pi}{4}, \frac{\pi}{2}] \quad [\frac{\pi}{2}, \frac{3\pi}{4}] \quad [\frac{3\pi}{4}, \pi] \]

1 Introduction
Let \( X \) be a curve of genus \( g \), \( \mathcal{A} = \{\Theta_i\}_{i \leq \frac{g}{2}} \) the Frobenius’s angles of \( X \).
Let \( q \) be a prime, \( r = \sqrt{q} \) and for \( n \in \mathbb{N}^* \), \( N_n \) the number of points over the field \( \mathbb{F}_q \). So:

\[ (1) \quad \forall n \in \mathbb{N}^*, \ N_n r^{-n} = r^n + r^{-n} - 2 \sum_{\Theta \in \mathcal{A}} \cos(n \Theta) \]

Let \((u_n)_{n \in \mathbb{N}}\) be a sequence. We define
\[ \psi : \mathbb{R} \rightarrow \mathbb{R} \quad \text{and} \quad f : \mathbb{R} \rightarrow \mathbb{R} \]
\[ x \mapsto \sum_{n \geq 1} u_n r^n \quad \Theta \mapsto u_0 + 2 \sum_{n \geq 1} u_n \cos(n \Theta) \]
So the sum of \( u_n \) (1) give, with \( N = N_1 \):

\[ (2) \quad N \psi(r^{-1}) + \sum_{n \geq 1} (N_n - N) u_n r^{-n} = u_0 g + \psi(r) + \psi(r^{-1}) - \sum_{\Theta \in \mathcal{A}} f(\Theta) \]

2 Finite case
We can use (2) with hypothesis
Classic case (Serre, 1983)

\[ u_0 = 1, \quad \forall n \geq 1, \ u_n \geq 0, \quad \forall \Theta \in \mathcal{A}, \ f(\Theta) \geq 0. \]
So we deduce: \( N < \frac{2}{\psi(r) - 1} + 1 + \frac{\psi(r)}{\psi(r) - 1} \)
Case (I)

\[ u_0 = 1, \quad \forall n \geq 1, \ u_n \geq 0, \quad \exists \Gamma \subset [0, \pi]/ \forall \Theta \in \Gamma, \ f(\Theta) \geq 0. \]
Every curve with (I) \( N > \frac{2}{\psi(r) - 1} + 1 + \frac{\psi(r)}{\psi(r) - 1} \) must have the sum \( \sum_{\Theta \in \mathcal{A}} f(\Theta) \) negative and so there is a Frobenius’s angle in \([0, \pi] - \Gamma \).
The function \( f(x) = \frac{2}{2-\sqrt{2}} \cos(2x) + \frac{2+\sqrt{2}}{2-\sqrt{2}} \cos(x) + 1 \) is not negative over \([0, \frac{\pi}{4}] \cup [\frac{\pi}{4}, \frac{3\pi}{4}] \). For \( q = 2 \), we find that a curve with \( N > 8 \frac{2\sqrt{2}}{1+2\sqrt{2}} g + 2\frac{\sqrt{2}}{1+2\sqrt{2}} \) has a Frobenius’s angle in \([0, \frac{3\pi}{4}] \). (This inequality applies to a genus between 2 and 28.)
Case (II)

\[ u_0 = 0, \quad \forall n \geq 1, \ u_n \geq 0, \quad \exists \Gamma \subset [0, \pi]/ \forall \Theta \in \Gamma, \ f(\Theta) \geq 0. \]
Every curve with (II) \( N > 1 + \frac{\psi(r)}{\psi(r) - 1} \) must have the sum \( \sum_{\Theta \in \mathcal{A}} f(\Theta) \) negative and so there is a Frobenius’s angle in \([0, \pi] - \Gamma \).
The function \( f(x) = \cos(2x) \) is not negative over \([0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, \pi]\). We find that a curve with \( N > 1 + n^4 \) has a Frobenius’s angle in \([\frac{\pi}{2}, \frac{3\pi}{2}]\). (There are curves with \( N > 1 + n^4 \).

Case (III)

\[ u_0 < 0, \quad \forall n \geq 1, \quad u_n \geq 0, \quad \exists \Gamma \subset [0, \pi] / \forall \Theta \in \Gamma, \quad f(\Theta) \geq 0. \]

If \( \psi(r^{-1}) > 0 \), then every curve with (III) \( N > \frac{n}{\psi(r^{-1})} g + 1 + \frac{\psi(r^{-1})}{\psi(r)} \) must have the sum \( \sum_{\Theta \in \Lambda} f(\Theta) \) negative, and so there is a Frobenius’s angle in \([0, \pi] - \Gamma\).

A general example

Look at:

\[ f : \mathbb{R} \rightarrow \mathbb{R} \]

\[ x \mapsto \prod_{k=0}^{n-1} \left( \cos(x) - \cos((2k+1)\pi) \right). \]

\( f \) is not negative in \([0, \pi] \cup [\frac{3\pi}{2}, \pi]\). We can write \( f(x) = \sum_{k=0}^{n} c_k \cos(kx) \), we have \( c_0 = c_1 = 0, \quad c_n = c_{2n}, \quad c_{n-1} = c_0, \quad \text{and} \quad \forall k \leq n, \quad c_k \geq 0. \)

So:

If \( N > 1 + r^n \), there is a Frobenius’s angle in \([\frac{\pi}{2}, \frac{3\pi}{2}]\).

3 Infinite case

Let \( B_n \) be the number of points of degree \( n \) on \( X \).

We suppose the following conditions:

\[ \exists (a, A, b, B, C) \in \mathbb{R} \times \mathbb{R}_+^4 / \forall n \in \mathbb{N}^*, \quad u_n \geq -AC^{-n}r^{-m} \quad \text{and} \quad \frac{B_n}{g} \leq Br^m \]

with \( \frac{r}{g^q} < 1 \). So we have:

\[ \sum_{n \geq 1} (N_n - N)u_n r^{-n} \geq ABgCr^{s+1} r^3 - 2Cr^{s+1} - C^2r^{2(a+1)} - 1 = b_i(g) \]

And if \( N > \frac{b_i(g)}{\psi(r^{-1})} + 1 + \frac{\psi(r)}{\psi(r)} \), there is an angle in \([0, \pi] - \Gamma\).

4 Open problems

1) Find other examples for the case (II) then for (III), and compare.
2) Find examples for the infinite case (good choice of \( a, A, b, B \)).
3) Find an upper bound of the length of an interval without Frobenius’s angle.
4) Find a lower bound of the maximum dimension of a component.

References

M. Tsfasman,
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