Designs and representation of the symmetric group

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1 Introduction

In this paper $\mathcal{F}_q$ is any set of q elements. For sake of simplicity we denote its elements $\{1, 2, \ldots, q\}$. Let n be an integer ; we set $X = \mathcal{F}_q^n$ and for x in X we define the *composition* of x : it is the q-tuple $s(x) = (s_1, \ldots, s_q)$ where $s_i \stackrel{def}{=} |\{j, x_j = i\}|$. Let s be a composition, that is to say a q-tuple which is the composition of some x in X; we set $X_s = \{x \in X, s(x) = s\}$.
We define and study a notion of designs for subsets of $\mathcal{F}_q^n$. In the case $q = 2$, our definition coincides with the usual notion of designs and in fact our definition is equivalent to the notion of strong colored designs of A. Bonnecaze, P. Solé, P. Udaya in *Tricolore 3-designs in type III codes*, to appear in *Discrete Math.*. We characterize them by orthogonality relations to isotypic components of the following representation of S_n:

$$[X_s] \simeq \mathrm{Ind}_H^G 1 \simeq \bigoplus_{\lambda \quad s} K_{\lambda s}[\lambda]$$

where H is a Young subgroup of S_n and the $[\lambda]$ are the Specht modules. As a matter of fact S_n acts transitively on the set X_s; it defines a non commutative association scheme (a coherent homogeneous configuration) and Theorem 1 shows that our designs are the generalized designs which corresponds to this association scheme.

2 Generalized designs

The *weight* of a word $x \in X$ is the number of entries of x which are not 1; all the x of composition s have of course the same weight; we note it $|s|$ and call it the *weight* of the composition : $|s| \stackrel{def}{=} s_2 + s_3 + \ldots + s_q$. We define the following order on X :

$$x \leq y \Leftrightarrow x_i = 1 \text{ or } y_i \text{ for } i = 1, \ldots, n.$$

We say that y *covers* x. Now we can define designs.

Definition 1 *A set of words $\mathcal{B}$ in $\mathcal{F}_q^n$ is a t-design if all the words in $\mathcal{B}$ have the same composition (let us say s) and if for each composition τ such that $|\tau| = t$, there is an integer λ_τ such that :*

$$\forall T \in X, \quad s(T) = \tau \Rightarrow |\{x \in \mathcal{B}, T \leq x\}| = \lambda_\tau.$$

We call *bleaching* the application from $\mathcal{F}_q$ to $\mathcal{F}_2$ that replaces any element of $\{3, 4, \ldots, q\}$ with "2": after "bleaching", a t-design is a classical t-design.

2.1 Main theorem

A design $\mathcal{B}$ is a subset of X_s. We define $f_\mathcal{B}$ as follows :

$$f_\mathcal{B} = \sum_{x \in \mathcal{B}} x \in \quad [X_s]$$

Then we have :

Theorem 1 *$\mathcal{B}$ is a t-design* $\Leftrightarrow f_\mathcal{B} \perp K_{\lambda s}[\lambda] \quad 0 < |\lambda| \leq t$

Note that with this theorem, it is obvious that a t-design is a $(t-1)$-design.