Designs in Grassmannian spaces and lattices

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This talk reports on results which are a common work with Renaud Coulangeon (Université de Bordeaux) and Gabriele Nebe (Ulm University).

The notion of designs originates in combinatorics with the combinatorial study of finite geometries. It was widely generalised to the setting of association schemes. In [2], the designs on manifolds where introduced with the case of the unit sphere (in which case designs are called spherical designs). Later, the same authors have considered all the rank one symmetric spaces.

The spherical designs have interesting applications to lattices, due to Boris Venkov. He introduces the notion of *strongly perfect lattices*, which are by definition the lattices which minimal vectors are a 4-*spherical design* in the sense of [2] (or equivalently a 4-design in the projective space, i.e. the Grassmann manigold $\mathcal{G}_{1,n}$, see below). The main point is that these lattices are a subclass of the extreme lattices, i.e. of the lattices on which the Hermite function is a local maximum, distinguished by a combinatorial property of their minimal vectors.

We introduce an analogous notion of t-design on the Grassmann manifold $\mathcal{G}_{m,n}$ of the m-subspaces of the Euclidean space ^{n}. To this aim we use the decomposition of the $O(n)$-module $L^2(\mathcal{G}_{m,n})$ of square integrable functions on $\mathcal{G}_{m,n}$ into a sum of irreducibles given by:

Theorem 0.1 *Let $m \leq \frac{n}{2}$. Then the $O(n)$-space $L^2(\mathcal{G}_{m,n})$ is isomorphic to*

$$L^2(\mathcal{G}_{m,n}) \simeq \bigoplus V_n^{\mu} \tag{1}$$

where the sum is over the μ of depth at most equal to m, with all the $\mu_i \equiv 0 \mod 2$.

where the irreducible modules V_n^{μ} are canoniquely associated to the partitions (see [3]). The zonal polynomials associated to this decomposition, which are polynomials in m variables (the space is of rank m), are explicitely computed in [4].

Definition 0.1 *Let $\mathcal{D}$ be a finite subset of $\mathcal{G}_{m,n}$, and let t be an even number. We say that $\mathcal{D}$ is a t-design if $\sum_{p\in\mathcal{D}} f(p) = 0$ for all $f \in H^{\mu}_{m,n}$ and for all μ with $2 \leq \deg(\mu) \leq t$.*

We give various criteria for t-designs on $\mathcal{G}_{m,n}$; in particular we characterize the finite subgroups of $O(n)$ which orbits on the grassmannians are always t-designs; this leads to a large family of examples, including the sections of certain important lattices like E_8 or the Leech lattice.

If the minimal m-sections of a lattice L are a 4-design, we call the lattice L *strongly m-perfect*. We show that the strongly m-perfect lattices are m-extreme, meaning that the Rankin function $\gamma_{n,m}$ has a local maximum on the lattice.

References

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