ON MAXIMAL CURVES AND ARTIN-SCHREIER EXTENSIONS

MIRIAM ABDÓN

An $\mathbf{F}_{q^2}$-maximal curve is a projective, geometrically irreducible, non-singular algebraic curve defined over the finite field $\mathbf{F}_{q^2}$ that attains the Hasse-Weil bound on the number of $\mathbf{F}_{q^2}$-rational points. Ihara [7] noticed that the genus g of a $\mathbf{F}_{q^2}$-maximal curve is upper bounded by $q(q-1)/2$. Stichtenoth and Rück [9] proved that the Hermitian curve, defined by $y^q + y = x^{q+1}$, is the unique $\mathbf{F}_{q^2}$-maximal curve (up to $\mathbf{F}_{q^2}$-isomorphism) of genus $q(q-1)/2$. Fuhrmann and Torres [5] showed that $g \leq (q-1)^2/4$ provided that $g < q(q-1)/2$, and there is a unique $\mathbf{F}_{q^2}$-maximal curve of genus $\lfloor (q-1)^2/4 \rfloor$, namely the quotient curve of the Hermitian curve by certain involutions (see [4], [2], [8]). Furthermore, $g \leq \lfloor (q^2-q+4)/6 \rfloor$ provided that $g < \lfloor (q-1)^2/4 \rfloor$, and the bound is sharp; however, it is not known whether or not an $\mathbf{F}_{q^2}$-maximal curve of genus $\lfloor (q^2-q+4)/6 \rfloor$ is unique (see [8]). Nonetheless, Cossidente et al. [3] observed the existence of two non-isomorphic $\mathbf{F}_{q^2}$-maximal curves of genus $(q-1)(q-3)/8$, and I constructed similar examples in characteristic two [1]. A plane model for the later examples is of Artin-Schreier type:

$$(*) \qquad \sum_{i=2}^{t} A_i z^{q/p} = x^{q+1} ,$$

where $q = p^t$, $A_i \in \mathbf{F}_q \subseteq \mathbf{F}_{q^2}$, $A_2 = 1$, and $A_t \neq 0$. For a numerical example take $q = 2^6$. Then both curves $z^{16} + z^8 + z^2 + z = x^{65}$ and $z^{16} + z^4 + z = x^{65}$ have the same genus, and they are $\mathbf{F}_{2^{12}}$-maximal since they are covered by the Hermitian curve via the morphisms $(x : y : 1) \mapsto (x : y^4 + y^2 + y : 1)$ and $(x : y : 1) \mapsto (x : y^4 + y : 1)$ respectively. In addition, the curves are not $\mathbf{F}_{2^{12}}$-isomorphic [1].

For a given $q = p^t$ with $t \geq 3$, in this work we plan to study the locus in $\mathbf{F}_{q^2}^{t-2}$ of the points $(A_3, \ldots, A_t)$ for which Eq. $(*)$ is the plane model of a $\mathbf{F}_{q^2}$-maximal curve. The starting point is the knowledge of the Weierstrass and Frobenius invariants (cf. [10]) of the linear series $|(q+1)P_0|$ on a $\mathbf{F}_{q^2}$-maximal curve with plane model $(*)$ (see [6]). Then the locus to be studied will be $\mathbf{F}_q$-rational points of a certain variety defined from the aforementioned invariants. In addition, since any curve covered by a maximal curve is also maximal, we can also study the maximality of $(*)$ by looking at the morphism $(x : y : 1) \mapsto (x : y^{p^2} + ay^p + by : 1)$ defined on the Hermitian curve. If $a, b \in \mathbf{F}_p$, then this morphism is onto $(*)$ if the following recursive relations hold:

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- For $i = 2, \dots, t-2$, $bA_i + aA_{i+1} + A_{i+2} = 0$;
- $bA_{t-1} + aA_t = 0$.

For example, if $q = p^3$, $p > 2$ and -3 is a square modulo p, then the above relations have a solution if and only if $b = a^2$, and either $a = -1$ or a is a root of $X^2 - X + 1 = 0$. In this case we obtain just one maximal curve. Now for $q = p^4$ the above relations have a solution if and only if $a(b^2 - 1) = 0$. If $a \neq 0$, we obtain a maximal curve if and only if -1 is a square in $\mathbf{F}_p$; otherwise, we obtain a maximal curve if and only if either 2 or -2 is a square in $\mathbf{F}_p$. In this case we obtain at least two non-isomorphic $\mathbf{F}_{q^2}$-maximal curves.

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Instituto de Matemática-UFF,
Coordenação de Pos-Graduação,
Rua Mário Santos Braga, s/n,
Campus do Valonguinho,
Niteroi-RJ, CEP 24.020-140, Brazil

E-mail address: `miriam@impa.br`