The Nyman-Beurling Equivalent Form for the Riemann Hypothesis

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1 Introduction

The Mellin transform defines a one-to-one correspondence between functions on $]0, +\infty[$ satisfying suitable growth conditions and holomorphic functions defined on some vertical strip of $\mathbb{C}$. To a certain extent, this correspondence establishes a dictionary between properties of functions and properties of their Mellin transforms.

In the case of the Riemann zeta-function, one may see $-\zeta(s)/s$ as a Mellin transform in the critical strip by means of the following formula:

$$\int_0^{+\infty} \rho \left( \frac{1}{t} \right) t^{-1} dt = -\frac{\zeta(s)}{s}, \quad 0 < \Re s < 1,$$

where $\rho(x)$ denotes the "fractional part" of $x$ (cf. [15], formula (2.1.5)). With this formula at hand, Nyman and Beurling had the idea that it should be possible to translate the Riemann hypothesis* into a property of $\rho$. This program was completed and yielded the following equivalent form for (RH), where $\rho_\alpha(t)$ denotes the function $\rho(\alpha/t) - \alpha \rho(1/t)$:

$$\text{span}_{L^2(0,1)} \{ \rho_\alpha, \ 0 < \alpha < 1 \} = L^2(0,1).$$

It is in fact possible to give an unconditional formula for the left-hand side of this equality in terms of $\zeta(s)$:

Theorem 0 (Bercovici and Foias, 1984 [5]) $Mf(s)$ being defined by (4), we have

$$\text{span}_{L^2(0,1)} \{ \rho_\alpha, \ 0 < \alpha < 1 \} = \{ f \in L^2(0,1), \ \frac{Mf(s)}{\zeta(s)} \text{ is holomorphic for } \Re s > \frac{1}{2} \}.$$ (3)

The main purpose of this text is to outline a proof of (3). We begin with some notations and remarks.

1. In formula (3) and the rest of this paper, we denote by $Mf(s)$ the Mellin transform of a function $f \in L^2(0,1)$, defined by

$$Mf(s) = \frac{1}{\sqrt{2\pi}} \int_0^1 f(t) t^{-1} dt.$$ (4)

Apart from the constant $1/\sqrt{2\pi}$, (4) is the usual definition of the Mellin transform, where $L^2(0,1)$ is considered as the subspace of $L^2(0, +\infty)$ of functions vanishing on $(1, +\infty)$. The constant factor $1/\sqrt{2\pi}$ makes $M$ a unitary operator (see remark 8 of section 3).

2. The holomorphy of $Mf(s)/\zeta(s)$ in the half-plane $\Re s > \frac{1}{2}$ is of course equivalent to the inclusion $Z(\zeta) \subset Z(Mf)$, where $Z(F)$ denotes the multiset\(^1\) of zeroes of $F(s)$ in the half-plane $\Re s > \frac{1}{2}$.

*Sometimes, we will denote the Riemann hypothesis by (RH).

\(^1\)The set of zeroes of $F(s)$ counting multiplicity.
3. It is easy to find functions $f$ in $L^2(0,1)$ whose Mellin transforms do not vanish in $\Re s > \frac{1}{2}$ (for instance the constant $1$). Thus the equivalence between (2) and the usual form of the Riemann hypothesis follows easily from theorem 0.

4. Taking orthogonals, we get the following equivalent form of (3), linking in a more direct way the functions $\rho_\alpha$ and the zeroes of $\zeta$ in $\Re s > \frac{1}{2}$:

$$\{\rho_\alpha, \ 0 < \alpha < 1\} = \text{span}_{L^2(0,1)} \{t \mapsto t^{\beta-1} \log^k t, \ \Re \beta > \frac{1}{2}, \ \zeta(\beta) = 0, \ 0 \leq k < \text{multiplicity of } \beta\}.$$

5. The proof of (3) is an application of the theory of the Hardy space $H^2$ of the half-plane $\Re s > \frac{1}{2}$. For this reason, we recall a few facts in the next section.

6. The proof of the equivalence of (2) and the Riemann hypothesis appears in Nyman’s thesis (theorem 7 of [13]) and was afterwards generalized by Beurling [6]. Formula (3) appears as corollary (2.2) in [5] (cf. also [12] for another point of view). We follow essentially the proof of Bercovici and Foias in [5] but our presentation is somehow different.

7. The criterion of Beurling (cf. [6], and [7] for a readable proof) states, for every $p$ such that $1 < p < +\infty$, the equivalence between the non vanishing of $\zeta(s)$ for $\Re s > 1/p$ and the density in $L^p(0,1)$ of the subspace generated by the $\rho_\alpha$, $0 < \alpha < 1$. Observing that $Mf$ is holomorphic for $\Re s > 1/p$ if $f \in L^p(0,1)$ we may ask if a generalized version of (3) holds, namely

Question 1 If $1 < p < +\infty$, is it true that

$$\text{span}_{L^p(0,1)} \{\rho_\alpha, \ 0 < \alpha < 1\} = \{f \in L^p(0,1), \ \frac{Mf(s)}{\zeta(s)} \text{is holomorphic for } \Re s > \frac{1}{p}\}?$$

As far as we know, this is an open question for every $p \neq 1$ and $p \neq 2$. Note that (RH) implies a positive answer for $1 \leq p \leq 2$ since, because of Beurling’s criterion, both spaces are then equal to $L^p(0,1)$.

2 Some facts related to the Hardy space $H^2(\Re s > \frac{1}{2})$

We\(^1\) denote here by $H^2 = H^2(\Re s > \frac{1}{2})$ the set of functions $F(s)$, holomorphic in the half-plane $\Re s > \frac{1}{2}$, with the following property:

$$\|F\|^2 := \sup_{s > 1/2} \int_{\Re z = x} |F(s)|^2 ds < +\infty. \quad (5)$$

This set is known as the Hardy space $H^2$ of the half-plane $\Re s > \frac{1}{2}$.

To each function $F \in H^2$, corresponds $F^* \in L^2(\mathbb{R})$, the boundary value function of $F$, defined for almost every $y \in \mathbb{R}$ by

$$F^*(y) = \lim_{x \to 1/2^-} F(x + iy),$$

and satisfying $\|F^*\|_{L^2(\mathbb{R})} = \|F\|$. Identifying $F$ and $F^*$, we may then see $H^2$ as a closed subspace of $L^2(\mathbb{R})$.

The main fact about the Hardy space is the existence of a factorization formula valid for every function in this space. We now describe this formula, and first the factors appearing in it, namely the inner and outer functions.

The inner functions are not elements of $H^2$, but rather of $H^\infty$, the set of bounded holomorphic functions in the half-plane $\Re s > 1/2$; they are the functions in $H^\infty$ whose boundary values have

\(^1\)For $p = 1$ also these two assertions are equivalent. In fact they are both true.

\(^2\)Strictly speaking, we did not find in the literature the introduction to the theory of $H^2(\Re s > \frac{1}{2})$ given in this section. But by conformal mapping between half-planes, it is an exercise to deduce it from the presentation of $H^2$ of the upper half-plane [6] or of the right half-plane [9].
modulus one almost everywhere (as for $H^2$, boundary values of functions in $H^\infty$ exist almost everywhere). It turns out that each inner function $\varphi$ can be written in a unique way as the product $\varphi = cB_S$, where $c$ is a constant, $|c| = 1$, $B$ is a Blaschke product and $S$ a singular inner function, defined as follows.

A **Blaschke product** is given by the formula

$$B(s) = B_Z(s) := \prod_{\beta \in Z} \left( \frac{|\beta(1-\beta)|}{\beta(1-\beta)} \frac{s-\beta}{s+\beta-1} \right),$$

where $Z$ is any multiset in the half-plane $\Re s > 1/2$ such that

$$\sum_{\beta \in Z} \frac{\Re(\beta - 1/2)}{|\beta|^2} < +\infty,$$

and with the convention $\frac{|\beta(1-\beta)|}{\beta(1-\beta)} = 1$ for $\beta = 1$.

A **singular inner function** is defined by the formula

$$S(s) = S_{\mu,v}(s) := \exp \left(-v \left(s - \frac{1}{2}\right) - \int_{-\infty}^{+\infty} \frac{2\pi s - (\frac{1}{2} + it)}{\frac{1}{4} + it - s} d\mu(t) \right),$$

where $v$ is any nonnegative real number, and

$\mu$ is a finite positive measure on the real line, singular to the Lebesgue measure.

An outer function is a function of $H^2$ defined for every function $\omega$ on the line such that

$$\omega \geq 0, \quad \omega \in L^2(\mathbb{R}), \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{\log \omega(t)}{1 + t^2} dt > -\infty,$$

by the formula

$$O_\omega(s) = \exp \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\pi s - (\frac{1}{2} + it) \log \omega(t)}{\frac{1}{4} + it - s} dt \right).$$

We are now in a position to state the fundamental

**Factorization formula:** Let $F$ be a function in $H^2$, $F \neq 0$. Then there exists a unique quintuple $(c, Z, \mu, v, \omega)$, with $c$ a complex number of modulus one, $Z$ a multiset satisfying (6), $\mu$ a measure satisfying (7), $v$ a nonnegative real number, and $\omega$ a function satisfying (8) such that

$$F = cB_SZ_{\mu,v}O_\omega.$$

Moreover, $Z$ is the multiset of zeroes of $F$ and $\omega = |F^*|$.

In the following, we shall use the notations $Z = Z(F)$, $\mu = \mu(F)$, $v = v(F)$. If $\mu(F) = 0$ and $v(F) = 0$, we shall say that $F(s)$ has no singular inner factor. An important property of $\mu$ is that its support is included in the set of $t$ such that $\frac{1}{2} + it$ is a singular point of $F$.

There is a natural notion of divisibility in the set of inner functions: $\varphi_1$ divides $\varphi_2$ if $\varphi_2/\varphi_1$ is inner. This is equivalent to

$$Z(\varphi_1) \subset Z(\varphi_2), \quad \mu(\varphi_1) \leq \mu(\varphi_2) \text{ and } v(\varphi_1) \leq v(\varphi_2).$$

Any family $(\varphi_\alpha)_{\alpha \in A}$ of inner functions has a greatest common divisor

$$\varphi = B_ZS_{\mu,v},$$

where

$$Z = \bigcap_{\alpha \in A} Z(\varphi_\alpha), \quad \mu = \inf_{\alpha \in A} \mu(\varphi_\alpha), \quad v = \inf_{\alpha \in A} v(\varphi_\alpha).$$

Let us specify what is meant by an intersection $Z = \bigcap_{\alpha \in A} Z_\alpha$ of multisets. If $s$ belongs to every $Z_\alpha$, then it belongs to $Z$ and its multiplicity in $Z$ is the infimum over $\alpha \in A$ of its multiplicity in $Z_\alpha$. 


3. Preparation for the proof of (3)

In order to show (3), we use on the one hand two important tools from the theory of $H^2$ (theorems 1 and 2 below) and on the other hand the computation of the Mellin transform of $\rho_\alpha$ and some technical information about it (lemmas 1 and 2).

Theorem 1 (Paley and Wiener 1934) The Mellin transform $M$ is unitary from $L^2(0,1)$ to $H^2$.

Theorem 2 (Beurling 1949, Lax 1959) Let $(F_\alpha)_{\alpha \in A}$ be a family of functions in $H^2$. Then the smallest closed subspace of $H^2$ invariant by multiplication by $e^{-\lambda s}$ for every $\lambda \geq 0$, and containing all the $F_\alpha$, $\alpha \in A$, is

$$B_Z S_{\mu,\nu} H^2,$$

where

$$Z = \bigcap_{\alpha \in A} Z(F_\alpha), \quad \mu = \inf_{\alpha \in A} \mu(F_\alpha), \quad \nu = \inf_{\alpha \in A} v(F_\alpha).$$

Lemma 1

$$M \rho_\alpha(s) = \frac{1}{\sqrt{2\pi}} (\alpha - \alpha^*) \frac{\zeta(s)}{s} (0 < \alpha < 1, \Re s > 0).$$

Lemma 2 For every $\alpha \in (0,1)$, the function $(\alpha - \alpha^*) \zeta(s)/s$ belongs to $H^2$ and has no singular inner factor. Moreover the space

$$\text{span}_{H^2}\{(\alpha - \alpha^*) \zeta(s)/s, 0 < \alpha < 1\}$$

is invariant by multiplication by $e^{-\lambda s}$ for every $\lambda \geq 0$.

These results call for some further comments.

8. We introduced the constant factor $1/\sqrt{2\pi}$ in the definition of the Mellin transform (4) for $M$ to be a unitary operator. More generally, thinking of $H^2$ as a closed subspace of $L^2(-\infty, +\infty)$ (via the boundary values), the Mellin transform $M$ appears as the restriction to $L^2(0,1)$ of the following multiplicative Fourier transform $F$ which is also unitary:

$$F : L^2(0, +\infty) \rightarrow L^2(-\infty, +\infty)$$

$$f \mapsto Ff(y) = \frac{1}{\sqrt{2\pi}} \lim_{T \to +\infty} \int_{1/T}^{T} f(t)e^{it - 1/2}dt.$$  

9. The usual form of Paley-Wiener's theorem is that the usual (additive) Fourier transform maps unitarily $L^2(0, +\infty)$ to $H^2(\Re s > 0)$ (see for instance theorem 11.9 of [8]). To deduce theorem 1 is just a matter of change of variables.

10. Lax's theorem, given in theorem 2, is a variant of the classical theorem of Beurling, describing the structure of the closed subspaces of $H^2(D)$ invariant by the shift operator (cf. [14], theorem 17.21).

11. In its usual statement (cf. [9] p. 107) Lax's theorem gives only the form $V = \varphi H^2$ for the closed invariant subspaces $V$ of $H^2$, where $\varphi$ is inner. But it is easy to see that $\varphi$ is necessarily the greatest common divisor of the inner factors of the functions in $V$, which gives our formulation.

12. The space $H^2$ can be thought of as a module over the integral ring $H^\infty$. It is easy to show that the closed subspaces of $H^2$ invariant by multiplication by $e^{-\lambda s}$ for every $\lambda \geq 0$ are

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\textsuperscript{1}The original paper [10] contains a much more general statement, dealing in fact with invariant subspaces of operator-valued (and not only scalar) functions in $L^2$ (and not only $H^2$).

\textsuperscript{2}In [9] (p. 107), Lax's theorem is given for subspaces of $H^2(\Re s > 0)$. It is obvious that the same statement holds with $H^2 = H^2(\Re s > 1/2)$.  

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13. Lemma 1, which follows easily from formula (1) is the core of the proof of (3). This lemma gives the fundamental relationship between \( \rho_\alpha \), \( \zeta \), and \( M \) from which (3) follows.

14. In lemma 2 the fact that \( (\alpha - \alpha^*)\zeta(s)/s \) belongs to \( H^2 \) follows immediately from lemma 1 and theorem 1. It has no singular inner factor because of the analytic continuation of \( \zeta \) through the line \( \Re s = 1/2 \) and the slow rate of convergence to 0 along the real half-line \( [1/2, +\infty) \) (see the proof of proposition 2.1 in [5]). Finally, the invariance of the span is due to the formula:

\[
e^{-\lambda^*}(\alpha - \alpha^*) = (ae^{-\lambda} - (ae^{-\lambda})^*) - \alpha(e^{-\lambda} - e^{-\lambda^*}).
\]

4 Proof of formula (3)

We first put

\[
Z = \bigcap_{0 < \alpha < 1} Z((\alpha - \alpha^*)\zeta(s)/s), \quad \mu = \inf_{0 < \alpha < 1} \mu((\alpha - \alpha^*)\zeta(s)/s) \quad \text{and} \quad v = \inf_{0 < \alpha < 1} v((\alpha - \alpha^*)\zeta(s)/s)
\]

and observe that \( Z = Z(\zeta) \) and, by lemma 2, \( \mu = 0 \) and \( v = 0 \).

Using successively theorem 1, lemma 1, lemma 2, theorem 2 and again lemma 2 for the first four equalities, we get:

\[
M(\text{span}_{\Re s(0, 1)} \{ \rho_\alpha, \ 0 < \alpha < 1 \}) = \text{span}_{H^2} \{ M\rho_\alpha, \ 0 < \alpha < 1 \} = \text{span}_{H^2} \{ (\alpha - \alpha^*)\zeta(s)/s, \ 0 < \alpha < 1 \} = B_2 S_{\mu, \alpha} H^2 = B_2(z) H^2 = \{ G \in H^2, \frac{G(s)}{\zeta(s)} \text{ is holomorphic for } \Re s > \frac{1}{2} \}.
\]

This last equality follows from the factorization theorem.

Comparing the first and last sets in this sequence of equalities concludes the proof of (3).

5 Remarks, open questions, and new results

Thus the Riemann hypothesis holds if and only if every function \( f \in L^2(0, 1) \) belongs to the \( L^2(0, 1) \)-closure of the vector space generated by the \( \rho_\alpha \), \( 0 < \alpha < 1 \). In fact, Beurling [6] noticed that (RH) already holds if the constant function 1 lies in this \( L^2(0, 1) \)-closure.

A minor variation of this criterion yields the following proposition, where \( \chi \) denotes the characteristic function of \( (0, 1) \).

**Proposition** The Riemann hypothesis is true if and only if

\[
\chi \in \text{span}_{L^1(0, +\infty)} \{ t \mapsto \rho(\alpha/t), \ 0 < \alpha < 1 \}.
\]

At first sight one may feel that it should not be too hard to find a sequence of linear combinations of \( \rho(\alpha/t) \) converging to \( \chi \) in \( L^2 \)-norm. This feeling is in a way strengthened by the formula

\[
\chi(t) = -\sum_{n \geq 1} \mu(n) \rho \left( \frac{1}{nt} \right),
\]

which holds for every positive \( t \), but in the sense of pointwise convergence. Here \( \mu \) denotes the Möbius function and the formula (9) follows from Möbius inversion and the prime number theorem\(^*\). Unfortunately, this feeling is, at least to a certain extent, a mirage!

\(^*\)The partial sums of the series in (9) thus appear as natural approximations to \( \chi \). Their study is undertaken in [1] and continued in [16], [11], [4], [2] and [3].
Anyhow, this necessary and sufficient condition leads to several questions. For instance, let us say that a closed subset $A$ of $[0, 1]$ is fundamental if the criterion remains valid if the condition $0 < \alpha < 1$ is replaced by $\alpha \in A$. Formula (9) motivates the following

**Question 2** Is $\{\frac{1}{n}, n \in \mathbb{N}, n \geq 1\}$ fundamental?

It is an open question. Note that, by use of the Plancherel theorem, it has the following equivalent formulation:

**Assume (RH).** Is it true that

$$\lim_{n \to +\infty} \inf_{(a_1, \ldots, a_n) \in \mathbb{C}^n} \int_{\mathbb{R}_{x=1/2}} |1 - \zeta(s)| \sum_{k=1}^{n} a_k k^{-s} \frac{|ds|}{|s|^2} = 0?$$

Thus formulated, this question leads naturally to other ones.

**Question 3** Is it true that

$$\lim_{n \to +\infty} \int_{\mathbb{R}_{x=1/2}} |1 - \zeta(s)| \sum_{k=1}^{n} \mu(k) k^{-s} \frac{|ds|}{|s|^2} = 0?$$

The answer is negative, by a recent work of Báez-Duarte containing interesting new ideas (cf. [2], theorem 2.2). If, however, the lim symbol is replaced by lim inf, the answer is not known.

Thinking of the Selberg weights $\mu(k)(1 - \log k / \log n)$, we may ask the more refined

**Question 4** Assume (RH). Is it true that

$$\lim_{n \to +\infty} \int_{\mathbb{R}_{x=1/2}} \left|1 - \zeta(s)\right| \sum_{k=1}^{n} \mu(k) \left(1 - \frac{\log k}{\log n}\right) k^{-s} \frac{|ds|}{|s|^2} = 0?$$

This is another open question.

We conclude this paper with a report on recent joint work with Báez-Duarte and Landreau. In order to tackle question 2, some numerical experiments were done, computing the first values of

$$d_n := \text{dist}_{L^2(0, +\infty)} \left(X, \left\{ \sum_{k=1}^{n} a_k \rho \left(\frac{1}{k^s}\right), (a_1, \ldots, a_n) \in \mathbb{C}^n \right\} \right).$$

Logarithmic rescaling and a statistical regression gave the following approximate formula for $d_n$, $n \leq 10000$,

$$d_n \approx \tilde{d}_n := \frac{0.213707}{\sqrt{\log n}}.$$ 

It led us to try, at least, to show that $d_n \gg \tilde{d}_n$. In fact, more is true.

**Theorem 3 (Báez-Duarte, Balazard, Landreau and Saias, 2000, [3])** When $n$ goes to infinity, one has:

$$d_n \geq \frac{c + o(1)}{\sqrt{\log n}},$$

where

$$c := \left(\sum_{|\beta|^2} \right)^{\frac{1}{2}},$$

the summation being extended over all non real zeroes $\beta$ of $\zeta$, each zero being counted only once, regardless of its multiplicity\(^{11}\).

\(^{11}\)In [3], only the zeroes on the critical line were counted in $c$. The result here is also correct because, in case (RH) is false, $c$ is nevertheless finite and $d_n$ is bounded below by a positive constant, a result stronger than (10).
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Of course, the value of $c$ may be computed numerically within any given accuracy but, to get a nice theoretical formula for it, we must assume (RH) and the simplicity of zeroes of $\zeta$, in which case one has:

$$c = \sqrt{2 + \gamma - \log(4\pi)}$$

where $\gamma$ denotes the Euler constant. Observe that

$$\sqrt{2 + \gamma - \log(4\pi)} = 0.214921...$$

This constant is near enough to the experimental constant 0.213707 to lead us to conjecture that the result of this last theorem is optimal.

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